BOUNDARY ELEMENTS THEORY AND APPLICATIONS



J.T. Katsikadelis

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BOUNDARY ELEMENTS: Theory and Applications

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John T. Katsikadelis

Department of Civil Engineering, National Technical University of Athens, Athens, Greece



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To my wife Efi for her loving patience and support



John T. Katsikadelis is a full professor at the Civil Engineering Department of the National Technical University of Athens (NTUA), Greece, the director of the Institute of Structural Analysis and Aseismic Research at NTUA and the professor of Structural Analysis at the School of Corps of Engineers of the Greek Army. He received his Civil Engineering Diploma and Doctor of Engineering Degree from NTUA and later a MSc degree and a second PhD degree in Applied Mechanics, from the Polytechnic University of New York. In 1970, he joined the faculty of NTUA where he has been teaching courses of statically determinate and indeterminate structures, matrix structural analysis, structural dynamics, plates

and shells, continuum mechanics and applied elasticity and boundary elements.

He has been involved in the design and construction of many civil engineering structures both as consultant and contractor. Dr. Katsikadelis maintains research interests in the areas of applied mechanics, structural engineering, and numerical methods, especially boundary elements. He has published more than 140 technical papers in the most reputable international journals and proceedings of international conferences, he has written chapters in edited books, text books and lecture notes on structural analysis.

He serves on the editorial boards of international journals and book series. He has been guest editor of special issues of Journals, member of scientific and advisory committees for many international conferences, where he has given plenary keynote lectures. He is the past president of the *Greek Association for Computational Mechanics* (affiliated to IACM), and the vice President of the *Hellenic Society for Theoretical and Applied Mechanics* (affiliated to IUTAM). He is also a fellow of the Wessex Institute (UK) and a founding member of the *International Society for Computational Engineering and Sciences* and a member of the *International Society Boundary Elements (ISBE), Hellenic Society of Steel Structures, Greek Society of Earthquake Engineering*, and *New York Academy of Sciences*. From 1989 until 1992, Dr. Katsikadelis had been the Director General of the Earthquake Planning and Protection Organization of Greece, Director of the European Center for Prevention and Forecasting Earthquakes of the Council of Europe, Permanent Correspondent of Greece in the Council of Europe for Major Natural and Technological Disasters.

Preface

The last three decades have been marked by the evolution of electronic computers and an enormous and wide-spread availability of computational power. This has boosted the development of computational methods and their application in engineering and in the analysis and design of *structures*, which extend from bridges to aircrafts and from machine elements to tunnels and the human body. New scientific *subfields* were generated in all engineering disciplines being described as "Computational", e.g. Computational Mechanics, Computational Fluid Mechanics, Computational Structural Analysis, Computational Structural Dynamics etc. The Finite Element Method (FEM) and the Boundary Element Method (BEM) are the most popular of the computational methods. While the FEM has been long established and is most well known in the engineering community, the BEM appeared later offering new computational capabilities with its effectiveness, accuracy and low computational cost.

Although the BEM is taught as a regular course at an ever increasing number of universities, there is a noticeable lack of a textbook which could help students as well as professional engineers to understand the method, the underlying theory and its application to engineering problems. An essential reason is that BEM courses are taught mainly as advanced graduate courses, and therefore much of the underlying fundamental knowledge of mathematics and mechanics is not covered in the respective undergraduate courses. Thus, the existing books on BEM are addressed rather to academia and researchers who, somehow, have already been exposed to the BEM than to students following a BEM course for the first time and engineers who are using boundary element software in industry.

This observation stimulated the author to write the book at hand. His research in the development of BEM during the last 25 years as well as the experience he acquired by teaching for many years the course of Boundary Elements at the Civil Engineering Department of the National Technical University of Athens, Greece, justify this endeavor. The author's ambition was to make BEM accessible to the student as well to the professional engineer. For this reason, his main task was to organize and present the material in such a way so that the book becomes "user-friendly" and easy to comprehend, taking into account only the mathematics and mechanics to which students have been exposed during their undergraduate studies. This effort led to an innovative, in many aspects, way of presenting BEM, including the derivation of fundamental solutions, the integral representation of the solutions and the boundary integral equations for various governing differential equations in a simple way minimizing a recourse to mathematics with which the student is not familiar. The indicial and tensorial notations, though they facilitate the authors' work and allow to borrow ready to use expressions from the literature, have been avoided in the present book. Nevertheless, all the necessary preliminary mathematical concepts have been included in order to make the book complete and self-sufficient.

In writing the book, topics requiring a detailed study for a deep and thorough understanding of the BEM, have been emphasized. These are:

- (i) The formulation of the physical problem.
- (ii) The formulation of the mathematical problem, which is expressed by the governing differential equations and the boundary conditions (boundary value problem).
- (iii) The conversion of the differential equations to boundary integral equations. This topic familiarizes the reader with special particular solutions, the socalled *fundamental solutions*, shows how they are utilized and helps to comprehend their singular behavior.
- (iv) The transformation of domain integrals to boundary line integrals or their elimination, in order to obtain pure boundary integral equations.
- (v) The numerical solution of the boundary integral equations. This topic, which covers a significant part of the book, deals with the numerical implementation of BEM rendering a powerful computational tool for solving realistic engineering problems. It contains the discretization of the boundary into elements, the modeling of its geometry, the approximation of the boundary quantities, as well as the techniques for the evaluation of regular and singular line integrals and in general the procedure for approximating the actual problem by a system of linear algebraic equations.
- (vi) A detailed description of the FORTRAN programs, which implement the numerical procedure for the various problems. The reader is provided with all the necessary information and the know-how so that he can write his own BEM-based computer programs for problems other than those included in the book.

(vii) The use of the aforementioned computer programs for the solution of representative problems and the study of the behavior of the corresponding physical system.

Throughout the book, every concept is followed by example problems, which have been worked out in detail and with all the necessary clarifications. Furthermore, each chapter of the book is enriched with problems-to-solve. These problems serve a threefold purpose. Some of them are simple and aim at applying and better understanding the presented theory, some others are more difficult and aim at extending the theory to special cases requiring a deeper understanding of the concepts, and others are small projects which serve the purpose of familiarizing the student with BEM programming and the programs contained in the CD-ROM.

The latter class of problems is very important as it helps students to comprehend the usefulness and effectiveness of the method by solving real-life engineering problems. Through these problems students realize that the BEM is a powerful computational tool and not an alternative theoretical approach for dealing with physical problems. My experience in teaching BEM shows that this is the students' most favorite type of problems. They are delighted to solve them, since they integrate their knowledge and make them feel confident in mastering BEM.

The CD-ROM which accompanies the book contains the source codes of all the computer programs developed in the book, so that the student or the engineer can use them for the solution of a broad class of problems. Among them are general potential problems, problems of torsion, thermal conductivity, deflection of membranes and plates, flow of incompressible fluids, flow through porous media, in isotropic or anisotropic, homogeneous or composite bodies, as well as plane elastostatic problems in simply or multiply connected domains. As one can readily find out from the variety of the applications, the book is useful for engineers of all disciplines. The author is hopeful that the present book will introduce the reader to BEM in an easy, smooth and pleasant way and also contribute to its dissemination as a modern robust computational tool for solving engineering problems.

In closing, the author would like to express his sincere thanks to his former student and Visiting Assistant Professor at Texas A&M University Dr. Filis Kokkinos for his carefully reading the manuscript and his suggestions for constructive changes. His critic and comments are greatly appreciated. Thanks also belong to my doctoral student Mr. G.C. Tsiatas, M.Sc., for checking the numerical results and the derivation of several expressions.

> J.T. KATSIKADELIS Athens January 2002

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Chapter 1

Introduction

1.1 Scope of the book

Since the Boundary Element Method (BEM) became an appealing area of research, twenty five years ago, several books have been published on this method [1-15]. These books present the theoretical background and the numerical application of this modern tool of analysis. Hence, it would be fair to pose the question "what is the purpose of writing one more book on the topic?". The answer is quite simple. All the existing books, although they describe comprehensively the method, for the most part they are written concisely. It could also be said that they are for academic use, and especially for the scientist that has already been exposed to the method and not for the student who studies the BEM for the first time. Moreover, since the BEM as a modern tool of solving engineering problems is intended for engineers, it must be presented in a way that can get across to them and bearing always in mind that extended utilization of advanced mathematics carries away authors in presenting the method rather as a subject of applied mathematics than a nice tool for solving engineering problems. For example, although use of tensors provides a concise and elegant formulation, it puts engineering students off. For this purpose, the book at hand presents the BEM and provides derivation of all the necessary equations by incorporating only fundamental concepts and basic knowledge from differential and integral calculus, and numerical integration. Since, the scope of this book is to present the BEM in a comprehensive way and not to study in-depth all its potentials, the application of the method is limited to simple problems. Some of them are boundary value problems governed by the Laplace or Poisson equation in two dimensions and plane elasticity problems. A considerable portion of the book is devoted to the numerical implementation of the method and its application to engineering problems. In all cases, computer programs are written in FORTRAN language. These programs, even though they solve important engineering problems, they are not professional but educational. Mainly, they present the logical steps required for their construction and they familiarize students with the development of a BEM software.

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The author anticipates that the book at hand will help students as well as field engineers to understand the BEM and apply it to problems they are faced with, either through the computer programs provided with the book or even their own. In addition, it is the author's strong belief that this book will contribute to a wider acceptance of the BEM as the most modern computational method.

1.2 Boundary Elements and Finite Elements

The Boundary Element Method (BEM) constitutes a technique for analyzing the behavior of mechanical systems and especially of engineering structures subjected to external loading. The term *loading* is used here in the general sense, referring to the external source which produces a non-zero field function that describes the response of the system (temperature field, displacement field, stress field, etc.), and it may be heat, surface tractions, body forces, or even non-homogeneous boundary conditions, e.g. support settlement.

Study of the behavior of structures is achieved today using computers. The reason is quite obvious, the low cost of the numerical versus the expensive experimental simulation. Numerical modeling can be used to study a wide variety of loadings and geometries of a structure and to determine the optimum design solution, before proceeding to its construction.

The method used for the numerical analysis of structures during the last 30 years is mainly the Finite Element Method (FEM). It is the method with which realistic problems of engineering are being solved, that is the analysis of structural elements of arbitrary geometry, arbitrary loading, variety of constitutive relations, with linear or non-linear behavior, in two or three dimensions. Justifiably, the FEM has been valued during the last 30 years as a modern computational tool.

A reasonable question to ask is "why do we need the BEM since we already have the FEM that solves engineering problems?". The answer is that a modeling with finite elements can be ineffective and laborious for certain classes of problems. So the FEM, despite the generality of its application in engineering problems, is not free of drawbacks. The most important of which are:

- (i) Discretization is over the entire domain occupied by the body. Hence, generation and inspection of the finite element mesh exhibit difficulty and are both laborious and time consuming, especially when the geometry of the body is not simple. For example, when there are holes, notches or corners, mesh refinement and high element density is required at these critical regions of large solution gradients (Fig. 1.1a).
- (ii) Modification of the discretized model to improve the accuracy of the solution or to reflect design changes can be difficult and requires a lot of effort and time.
- (iii) For infinite domains, e.g. half-space or the complementary domain to a finite one, fabrication of fictitious closed boundaries is required in order to apply

the FEM. This reduces the accuracy and some times may result in spurious or incorrect solutions.

- (iv) For problems described by differential equations of fourth or higher order (i.e., plate equations, or shell equations of sixth, eighth or higher order), the conformity requirements demand such a tedious job that FEM may become impractical.
- (v) Although the FEM computes accurately the field function, which is the unknown of the problem, it is ineffective in determining its derivatives. The accuracy drops considerably in areas of large gradients.



(a) FEM



Figure 1.1 (a) Domain (FEM) and (b) boundary (BEM) discretization.

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Drawbacks (i) and (ii) can be overcome by using updated releases of advanced professional finite element software, such as NASTRAN, which are equipped with automatic and adaptable mesh generators. Essentially, the task of generating a finite element mesh is a hard geometric problem, and in several cases, it may prove to be far more difficult than the physical problem which is to be solved by the FEM. The new disadvantage, though, is that the codes for creating FEM meshes are closed, and any effort to produce them requires specialized knowledge from a different scientific area. An attempt, by the engineer, to learn the necessary material will only distract him from his original goal, which is to solve a physical problem.

On the contrary, the Boundary Element Method possesses many advantages, the most important of which are:

- (i) Discretization is only over the boundary of the body, making the numerical modeling with the BEM easy (see Fig. 1.1b) and reducing the number of unknowns by one order. Thus, a remodeling to reflect design changes becomes simple.
- (ii) For infinite domains, the problem is formulated simply as an exterior one. Apparently, the fundamental solution has to satisfy some conditions at infinity, such as Sommerfeld's radiation condition for problems in dynamics. In this manner, computer programs developed for finite domains can be used, with just few modifications, to solve problems in infinite domains. This is not possible with the FEM.
- (iii) The method is particularly effective in computing the derivatives of the field function (e.g., fluxes, strains, stresses, moments). It can easily handle concentrated forces and moments, either inside the domain or on the boundary.
- (iv) The BEM allows evaluation of the solution and its derivatives at any point of the domain of the problem and at any instant in time. This is feasible because the method uses an integral representation of the solution as a continuous mathematical expression, which can be differentiated and utilized as a mathematical formula. This is impossible with the FEM, since the solution is obtained only at the nodal points.
- (v) The method is well suited for solving problems in domains with geometric peculiarities, such as cracks.

At its current stage of development, the BEM exhibits the following main disadvantages:

(i) Application of the BEM requires the so-called *fundamental solution*. The method cannot be used for problems whose fundamental solution is either not known or cannot be determined. Such are, for example, problems described by differential equations with variable coefficients. The method is obviously not applicable to non-linear problems for which the principle of superposition does not hold. In this case, a BEM model produces domain in-

tegrals that can be computed by discretizing the domain, but this, of course, spoils the pure boundary character of the method. During the last years, intense research has been conducted in an effort to overcome the aforementioned disadvantages.

(ii) The numerical implementation of the BEM results in systems of linear algebraic equations whose coefficient matrices are fully populated and non-symmetric. In a FEM model, however, the corresponding matrices have some very nice properties, they are banded and symmetric. This drawback of the BEM is counterbalanced by the much smaller dimensions of its matrices. The general format of the coefficient matrices for a FEM and BEM model is shown graphically in Fig. 1.2.



Figure 1.2 Coefficient matrices for FEM and BEM.

1.3 Historical development of the BEM

Until the beginning of the eighties, the BEM was known as Boundary Integral Equation Method (BIEM). As a method for solving problems of mathematical physics has its origin in the work of G. Green [16]. He formulated, in 1828, the integral representation of the solution for the Dirichlet and Neumann problems of the Laplace equation by introducing the so-called Green's function for these problems. In 1872, Betti [17] presented a general method for integrating the equations of elasticity and deriving their solution in integral form. Basically, this may be regarded as a direct extension of Green's approach to the Navier equations of elasticity. In 1885, Somigliana [18] used Betti's reciprocal theorem to derive the integral representation of the solution for the elasticity problem, including in its expression the body forces, the boundary displacements and the tractions.

The fatherhood, however, of the Boundary Element Method could be attributed to Fredholm. At the beginning of the twentieth century, he was the first one to use singular boundary integral equations in order to find the unknown boundary quantities for problems of potential theory [19]. In fact, the method was employed as a

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mathematical tool to determine the necessary boundary conditions for a well-posed problem of mathematical physics, and not as a method to solve the problem. This is quite reasonable, because it was, and still is, not possible to find the analytic solution of the derived singular integral equations. In the aforementioned methods, the unknown boundary quantities have a direct physical or geometrical meaning and for this reason they are referred to as *direct* BEM. In addition to these methods, there were also other BEM formulations developed, in which the unknown boundary quantities have no direct physical or geometrical meaning, and thereafter they are given the name *indirect* BEM [20, 21, 22]. A detailed review of these methods may be found in [23]. Sherman [24, 25], Mikhlin [26] and Muskhelishvili [27] used complex functions to develop boundary integral equation methods for the solution of plane elasticity problems.

Closed form solutions of integral equations were only derived for some domains with very simple boundary geometry. Unfortunately, the work of Fredholm predated the computers, which could make his ideas practical For this reason, the Boundary Integral Equation Method was neglected until the end of the fifties. Then, with the advent of computers, the method came back to the spotlight as an appealing numerical method for solving engineering problems. Numerical methods were developed for the solution of boundary integral equations and difficult physical problems of complex boundary geometry, which could not be tackled by other methods, were solved for the first time by the BIEM. The first works that laid the foundation of BEM as a computational technique appeared in the early sixties. Jaswon [28] and Symm [29] used Fredholm's equations to solve some two-dimensional problems of potential theory [30, 31]. The merits of BEM, which were listed in the previous section, attracted researchers and motivated them to further develop the method. Rizzo [32] and Cruse [33] applied the method to two-and threedimensional elasticity problems, respectively. Rizzo and Shippy [34] extended the method to anisotropic elasticity, while Cruse and Rizzo [35] solved the elastodynamic problem. Ignaczak and Nowacki [36] expressed the integral equations of thermoelasticity and Mendelson [37] studied problems of elastoplastic torsion.

All the aforementioned problems are governed by second order partial differential equations. Another group of problems are those described by the biharmonic equation. In this case, the integral representation of the solution was derived from the Rayleigh-Green identity [38], and the approach was applied to plate bending and plane elasticity, with the latter being formulated in terms of Airy's stress function. The formulation consists of two boundary integral equations, one for each of the unknown boundary quantities. The first one arises from the boundary character of the integral representation of the field function, while the second is obtained from the integral representation either of the Laplacian of the field function or its derivative along the normal to the boundary. The second approach presented by Katsikadelis et al. [39] became the prevailing one and was adopted later by Bezine [40] and Stern [41] to solve the plate bending problem. An extended and detailed presentation of the plate bending problems that have been analyzed by the BEM can be found in Ref. [42]. Already in the late eighties, one could find numerous

publications in the literature, where the BEM was applied to a wide variety of engineering problems. Among them are static and dynamic, linear or non-linear problems of elasticity, of plates and shells, problems of elastodynamics, wave and earthquake engineering, geomechanics and foundation engineering, soil-structure interaction, fluid-structure interaction, fluid dynamics, unilateral contact, fracture mechanics, electricity and electromagnetism, heat conduction, acoustics, aerodynamics, corrosion, optimization, sensitivity analysis, inverse problems, problems of system identification, etc. It could be said that today the BEM has matured and become a powerful method for the analysis of engineering problems, and an alternative to the domain methods. The method has been established by the name BEM (Boundary Element Method), which is attributed to the approach used to solve the boundary integral equations (i.e., discretization of the boundary into elements). Software based on the BEM has been developed for computers of simple or parallel architecture, along with professional high performance packages, like BEASY [43]. In 1978, C. Brebbia organized the first international conference on BEM, and since then conferences on BEM are organized yearly by the International Society for Boundary Elements (ISBE) and the International Association for Boundary Elements (IABEM). Furthermore, all conferences on computational mechanics devote sessions to the BEM. A detailed review of this enormous work would occupy a lot of space in this book and, of course, it is beyond its scope. However, interested readers are referred to literature review articles [44, 45], to the proceedings of the above conferences (BEM, IABEM) and to the numerous publications of the Computational Mechanics Publications, Southampton.

New developments in BEM aim at overcoming any drawbacks of the method. They deal with complicated time-dependent problems, linear problems for which the fundamental solution is not known, and also non-linear problems. For all these types of problems the resulting integral solution involves domain integrals, which complicate the application of the method. The most promising techniques that successfully overcome most of the difficulties and at the same time preserve the purely boundary character of the BEM, are the Dual Reciprocity Method (DRM) [46], which has, however, some limitations, and the Analog Equation Method (AEM) [47, 48]. The latter is general and relieved of DRM's limitations.

1.4 Structure of the book

As it was mentioned in Section 1.1, the scope of this book is to make the BEM comprehensible to the engineering student. For this purpose, its application will be limited to the simpler but most representative problems. On these grounds, the book contains including the introduction, seven chapters and three appendices. Every chapter is followed by the pertinent bibliography and recommended references for further study. Several exercises have been included at the end of each chapter in order to help the reader practice the concepts studied in the book.

Chapter 2 presents some preliminary mathematical concepts that are necessary for developing the BEM. These concepts are the divergence theorem of Gauss,

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Green's reciprocal identity (Green's second theorem) and the definition of the delta function along with its properties.

In Chapter 3, the direct BEM is developed for some problems of potential theory, that is boundary value problems for the two-dimensional Laplace and Poisson equations. It is extended also to the case of general second order partial differential equations with constant coefficients, which govern the homogeneous orthotropic, or generally, anisotropic bodies.

Chapter 4 describes the numerical implementation of BEM and the numerical solution of the singular boundary integral equations. For educational purposes, the solution is derived only for constant boundary elements and a computer program has been developed in FORTRAN language. The program is explained thoroughly and its structure is given in a systematic way, so that the student can become acquainted with the logic of writing BEM software. The method is also applied to domains containing holes and another computer program is provided for this case. Finally, one will find the method of subregions (subdivision of the original domain) as it applies to the BEM.

Chapter 5 is devoted to the boundary element technology. The singular integral equations are integrated numerically using boundary elements. The elements may be subparametric, isoparametric or superparametric and emphasis is put on the linear and parabolic elements. A considerable portion of the chapter is devoted to the evaluation of the singular and hypersingular integrals.

Chapter 6 presents applications of the BEM to engineering problems that can be reduced to boundary value problems for the Laplace or Poisson equation. In particular, the BEM is applied to the Saint-Venant torsion problem for isotropic and anisotropic materials, to the bending of membranes and of simply supported plates, to heat conduction and to the irrotational flow of incompressible fluids. For each of these problems, the reader is provided with a computer program and representative examples.

Chapter 7 studies the plane elasticity problem. The fundamental solution and the corresponding boundary integral equations are derived in a simple way so that the student will be able to follow all the steps and understand the subject. A computer program has been written also for this problem and several numerical applications are presented with intend to familiarize the student with the use of the program, on one hand, and to demonstrate the effectiveness of the method, on the other.

Appendix A lists useful relations, which facilitate the differentiation of the kernels found in the integral equations. Appendix B presents the Gaussian quadrature (numerical integration) for regular and singular integrals in one and two dimensions. Finally, Appendix C provides answers and/or hints to selected problems from those found at the end of the six chapters.

1.5 CD-ROM contents

The book is accompanied by a CD-ROM containing the computer programs whose code is presented in the chapters of the present book. Specifically, the following programs can be found in the CD-ROM.

- 1. LABECON.FOR, RECT-1.FOR and ELLIPSE-1.FOR. The first one solves the Laplace equation using constant elements, while the other two form the data files for rectangular and elliptic domains.
- 2. LABECONMU.FOR and RECT-2.FOR. The first program solves the Laplace equation for domains with holes and the second one prepares the data file for multiply connected domains (Example 4.3).
- 3. TORSCON.FOR, RECT-3.FOR and ELLIPSE-3.FOR. The first one solves the torsion problem and the other two create the data files for rectangular and elliptic cross-sections.
- 4. FLUIDCON.FOR. It analyzes the irrotational flow of incompressible fluids.
- 5. ELBECON.FOR, RECT-4.FOR and RECTEL-MU.FOR. The first one solves the plane elasticity problem (plane strain and plane stress). The second one constructs the data file for rectangular domains, while the third one forms the data file for domains with multiple boundaries (Example 7.3).

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Chapter 2

Preliminary Mathematical Concepts

2.1 Introduction

In this chapter, some mathematical relations are presented which are required for the development and understanding of the boundary element method (BEM). Although these relations could have been included in an appendix, they are placed here to show the reader their important role in the theoretical foundation and development of the BEM. They will be used many a time and oft in the book and particularly for the transformation of the differential equations, which govern the response of physical systems within a domain, into integral equations on the boundary. The understanding of these mathematical concepts will give the reader a feeling of confidence for their subsequent use.

2.2 The Gauss-Green theorem

The Gauss-Green theorem is a fundamental identity, which relates the integral of the derivative of a function over a domain Ω to the integral of that function on its the boundary Γ . The domain may be two- or three-dimensional. For simplicity of presentation, this relationship is derived for the two-dimensional case. Consider the plane domain Ω bounded by the curve Γ . We shall work first with the derivative of a function f = f(x, y) with respect to x. The integral over Ω may be written as a double integral, for which the integration is carried out first with respect to x and then with respect to y. Thus, we can write

$$\int_{\Omega} \frac{\partial f}{\partial x} d\Omega = \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx \right) dy = \int_{y_1}^{y_2} \left\{ f(x_2, y) - f(x_1, y) \right\} dy$$
(2.1)

where

$$x_1 = x_1(y)$$
 and $x_2 = x_2(y)$ (2.2)



Figure 2.1 Integration over a plane domain Ω bounded by a curve Γ .

From the detail of Fig. 2.1, we have

$$\frac{dy}{ds} = \cos \alpha = n_x \implies dy = n_x \, ds \tag{2.3a}$$
$$-\frac{dx}{ds} = \sin \alpha = n_y \implies dx = -n_y \, ds \tag{2.3b}$$

where n_x and n_y are the components of the unit vector **n**, which is normal to the boundary Γ . The negative sign in Eq. (2.3b) is due to the fact that the dx and the $\sin \alpha$ have opposite signs when the angle α is measured in the counter-clockwise sense with respect to the positive x-direction (see detail in Fig. 2.1).

Consequently, Eq. (2.1) becomes

$$\int_{y_1}^{y_2} \left\{ f(x_2, y) - f(x_1, y) \right\} dy = \int_{s_2} f(x_2, y) n_x \, ds - \int_{s_1} f(x_1, y) n_x \, ds \tag{2.4}$$

In the previous expression the integration on s_1 is performed in the negative direction (clockwise) when y varies from y_1 to y_2 . Using uniform direction for the integration over s, both terms in Eq. (2.4) can be combined in a single expression

$$\int_{\Omega} \frac{\partial f}{\partial x} d\Omega = \int_{\Gamma} f \, n_x \, ds \tag{2.5}$$

Interchanging x with y in Eq. (2.5), we obtain

$$\int_{\Omega} \frac{\partial f}{\partial y} \, d\Omega = \int_{\Gamma} f \, n_y \, ds \tag{2.6}$$

If g is another function of x and y, then Eqs. (2.5) and (2.6) result in

$$\int_{\Omega} \frac{\partial (fg)}{\partial x} d\Omega = \int_{\Gamma} fg \, n_x \, ds = \int_{\Omega} g \, \frac{\partial f}{\partial x} d\Omega + \int_{\Omega} f \, \frac{\partial g}{\partial x} d\Omega \quad \Rightarrow$$

$$\int_{\Omega} g \, \frac{\partial f}{\partial x} d\Omega = -\int_{\Omega} f \, \frac{\partial g}{\partial x} d\Omega + \int_{\Gamma} fg \, n_x \, ds \qquad (2.7)$$

$$\int_{\Omega} \frac{\partial (fg)}{\partial y} d\Omega = \int_{\Gamma} fg \, n_y \, ds = \int_{\Omega} g \, \frac{\partial f}{\partial y} d\Omega + \int_{\Omega} f \, \frac{\partial g}{\partial y} d\Omega \quad \Rightarrow$$

$$\int_{\Omega} g \frac{\partial f}{\partial y} d\Omega = -\int_{\Omega} f \frac{\partial g}{\partial y} d\Omega + \int_{\Gamma} fg \, n_y \, ds \tag{2.8}$$

Equations (2.7) and (2.8) state the integration by parts in two dimensions and are known as the *Gauss-Green theorem*.

2.3 The divergence theorem of Gauss

The divergence theorem results readily as an application of the Gauss-Green theorem. Consider the vector field $\mathbf{u} = u \mathbf{i} + v \mathbf{j}$, where \mathbf{i} , \mathbf{j} denote the unit vectors along the x and y axes and u = u(x, y), v = v(x, y) its components. Applying Eq. (2.5) for f = u and Eq. (2.6) for f = v and adding, yields

$$\int_{\Omega} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) d\Omega = \int_{\Gamma} \left(u \, n_x + v \, n_y \right) ds \tag{2.9}$$

If the coordinates x and y are represented by x_1 and x_2 , respectively, then the components of the vector field \mathbf{u} are denoted by u_i (i = 1, 2) and those of the normal vector \mathbf{n} by n_i . Therefore, Eq. (2.9) can be written as

$$\int_{\Omega} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) d\Omega = \int_{\Gamma} (u_1 n_1 + u_2 n_2) ds$$
(2.10)

or using the summation convention

$$\int_{\Omega} \frac{\partial u_i}{\partial x_i} d\Omega = \int_{\Gamma} u_i \, n_i \, ds \qquad (i = 1, 2)$$
(2.11)

Equations (2.9), (2.10) and (2.11) can also be written using vector notation as

$$\int_{\Omega} \nabla \cdot \mathbf{u} \ d\Omega = \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \ ds \tag{2.12}$$

in which the symbolic vector ∇ is defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} = \mathbf{i}_1\frac{\partial}{\partial x_1} + \mathbf{i}_2\frac{\partial}{\partial x_2}$$
(2.13)

and represents the differential operator that produces the gradient of a scalar field.

The quantity $\nabla \cdot \mathbf{u}$, i.e. the dot product of the vectors ∇ and \mathbf{u} , is referred to as the *divergence* of a vector field \mathbf{u} at a point inside the domain Ω , whereas the quantity $\mathbf{u} \cdot \mathbf{n}$ is referred to as the *flux* of the vector field at a point on the boundary Γ . The latter dot product expresses the projection of \mathbf{u} in the direction of \mathbf{n} . Eq. (2.12) relates the total divergence to the total flux of a vector field and it is known as the *divergence theorem of Gauss*. It is one of the most important theorems of integral calculus.

2.4 Green's second identity

Consider the functions u = u(x, y) and v = v(x, y) which are twice continuously differentiable in Ω and once on Γ . Applying Eq. (2.7) for g = v, $f = \frac{\partial u}{\partial x}$ and Eq. (2.8) for g = v, $f = \frac{\partial u}{\partial y}$ and adding the resulting equations, we arrive at the following

$$\int_{\Omega} v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega = -\int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) d\Omega + \int_{\Gamma} v \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) ds$$
(2.14)

Similarly, applying Eq. (2.7) for g = u, $f = \frac{\partial v}{\partial x}$ and Eq. (2.8) for g = u, $f = \frac{\partial v}{\partial y}$ and adding the resulting equations, we obtain

$$\int_{\Omega} u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) d\Omega = -\int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) d\Omega + \int_{\Gamma} u \left(\frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y \right) ds$$
(2.15)

Subtracting Eq. (2.15) from Eq. (2.14) yields

$$\int_{\Omega} \left(v \nabla^2 u - u \nabla^2 v \right) d\Omega = \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$
(2.16)

where ∇^2 is known as the *Laplace operator* or *harmonic operator* and it is defined as

$$\nabla^2 \equiv \nabla \cdot \nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
(2.17)

while

$$\frac{\partial}{\partial n} \equiv \mathbf{n} \cdot \nabla = \left(n_x \, \mathbf{i} + n_y \, \mathbf{j} \right) \cdot \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} \right) = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}$$
(2.18)

is the operator that produces the derivative of a scalar function in the direction of n. Equation (2.16) is known as *Green's second identity* for the harmonic operator or *Green's reciprocal identity*.

2.5 The adjoint operator

Consider the complete second order differential equation with variable coefficients

$$L(u) = A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$$
(2.19)

where A, B, ..., F are given functions of x and y in Ω . Multiplying Eq. (2.19) by a function v = v(x, y) and integrating over the domain, we have

$$\int_{\Omega} v L(u) \, d\Omega = 0 \tag{2.20}$$

Let us assume that v is twice continuously differentiable in Ω and once on Γ . Subsequently, integrating Eq. (2.20) by parts repeatedly until all derivatives of u are eliminated and incorporating Eqs. (2.7) and (2.8), we arrive at

$$\int_{\Omega} \left\{ v L(u) - u L^{*}(v) \right\} d\Omega = \int_{\Gamma} \left(X n_{x} + Y n_{y} \right) ds$$
(2.21)

where

$$L^{*}(v) = \frac{\partial^{2}(Av)}{\partial x^{2}} + 2\frac{\partial^{2}(Bv)}{\partial x \partial y} + \frac{\partial^{2}(Cv)}{\partial y^{2}} - \frac{\partial(Dv)}{\partial x} - \frac{\partial(Ev)}{\partial y} + Fv$$
(2.22)

$$X = A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv$$
(2.23)

$$Y = B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv$$
(2.24)

The differential operator $L^*()$ defined in Eq. (2.22) is referred to as the *adjoint* operator of L(). Equation (2.21) is the general form of the Green's second identity (2.16), which results readily as a special case of Eq. (2.21) for A = C = 1 and B = D = E = 0. The case $F \neq 0$ does not affect Eq. (2.16) as it is equivalent to adding and subtracting the term Fuv in the integral of the left hand side of this equation. When A, B, C are constants and D = E = 0, Eq. (2.22) becomes

$$L^{*}(v) = A \frac{\partial^{2} v}{\partial x^{2}} + 2B \frac{\partial^{2} v}{\partial y^{2}} + C \frac{\partial^{2} v}{\partial y^{2}} + Fv$$
(2.25)

Namely, the operator $L^*()$ is identical to L() and in this case L() is called *self-adjoint*.

The character of the solution of Eq. (2.19), as well as the type of the problem to be solved, depends on the quantity $\Delta = B^2 - AC$. We distinguish three types of equations:

- (a) Elliptic type, for $\Delta < 0$
- (b) Parabolic type, for $\Delta = 0$
- (c) Hyperbolic type, for $\Delta > 0$

2.6 The Dirac delta function

In problems of solid mechanics, we often come across *concentrated loads*, that is actions that are applied to a very small region, theoretically at a point, of the space or of the time.





Figure 2.2 Circular disc loaded by vertical concentrated unit load F.

Figure 2.3 Force distribution on the boundary of the body.

For example, consider the plane elastic body Δ_1 having constant thickness h and occupying the half-plane $-\infty < x < +\infty$, $y \ge 0$ (Fig. 2.2). A circular disc Δ_2 of the same thickness h and radius R is in contact with the semi-infinite body at point (x, y) = (0, 0) of its free boundary. The circular disc is loaded by a vertical concentrated unit load F as shown in Fig. 2.2. Because of the deformation of the elastic bodies the contact does not take place at a point, but it is extended over a small region. The function f(x) represents the distribution per unit length of the force applied on the boundary of the body Δ_1 . This function, which most probably

has the shape shown in Fig. 2.3, is not a priori known. However, we know that it is sufficiently *concentrated* and

$$\int_{-\infty}^{+\infty} f(x) dx = 1 \tag{2.26}$$

which expresses that the total force applied on the body Δ_1 is equal to unity. If we skip the problem of determining analytically the function f, we may assume *a priori* a prescribed shape for this function as

$$f_k(x) = \begin{cases} k/2, & |x| < \frac{1}{k} \\ 0, & |x| \ge \frac{1}{k} \end{cases}$$
(2.27)

or

$$f_k(x) = \frac{k}{\pi (1 + k^2 x^2)}$$
(2.28)

where k is a positive number.

Figures 2.4a and 2.4b show that both functions f_k defined in Eqs. (2.27) and (2.28), respectively, become sufficiently *concentrated* for large values of k. Moreover, they satisfy Eq. (2.26), which means that they are statically equivalent to f.



Figure 2.4 Functions f_k sufficiently concentrated for large values of k.

The distribution of f(x) becomes more "concentrated" as the bodies become less deformable. This is expressed by increasing the values of k in Eqs. (2.27) and (2.28). The limiting case where the bodies are rigid is expressed by $k \to \infty$. This

produces a fictitious distribution of the unit force per unit length, which is denoted by $\delta(x)$ and is defined as

$$\delta(x) = \lim_{k \to \infty} f_k(x) \tag{2.29}$$

where $f_k(x)$ is the function defined in Eq. (2.27) or (2.28). The function $\delta(x)$ is known as *delta function* or *Dirac delta function*.

In mathematics, the delta function is treated in the theory of generalized functions. We give below the definition of the delta function as a generalized function of one and two variables, and we mention some of its properties used in developing the BEM.

The one-dimensional delta function is defined by the relation

$$\int_{-\infty}^{+\infty} \delta(x) h(x) \, dx = h(0) \tag{2.30}$$

for a point source applied at the position x = 0 or by the relation

$$\int_{-\infty}^{+\infty} \delta(x - x_o) h(x) \, dx = h(x_o) \tag{2.31}$$

for a point source applied at $x = x_o$. The function h(x) is continuous in a finite interval containing the source point x = 0 or $x = x_o$, and has zero value outside the interval. The one-dimensional delta function can also be described by the relations

$$\delta(x) = \begin{cases} 0, & x \neq 0\\ \infty, & x = 0 \end{cases}$$
(2.32a)

$$\int_{-\infty}^{+\infty} \delta(x) \, dx = \int_{-\varepsilon}^{\varepsilon} \delta(x) \, dx = 1 \tag{2.32b}$$

where ε is a positive number. According to this definition, the function $\delta(x)$ has zero value everywhere except at point x = 0, where it becomes infinite, and satisfies Eq. (2.32b).

Equation (2.30) may be obtained from Eq. (2.32b) by applying the mean value theorem of integral calculus. Referring to Fig. 2.4a and choosing $\varepsilon = 1/k$, we obtain

$$\int_{-\infty}^{\infty} h(x)\,\delta(x)\,dx = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} h(x)\,f_k(x)\,dx = \lim_{\varepsilon \to 0} \left[h(x^*)\frac{1}{2\varepsilon}\right] 2\varepsilon$$
$$= \lim_{\varepsilon \to 0} \left[h(x^*)\right] = h(0)$$

The Dirac delta function $\delta(Q - Q_o)$ in two dimensions is defined as

$$\int_{\Omega} \delta(Q - Q_o) h(Q) \, d\Omega_Q = h(Q_o), \qquad Q(x, y), \, Q_o(x_o, y_o) \in \Omega \tag{2.33}$$

for an arbitrary function h(Q), which is continuous in the domain Ω containing the point $Q_o(x_o, y_o)$. The two-dimensional delta function may also be described by

$$\delta(Q-Q_o) = \begin{cases} 0, & Q \neq Q_o \\ \infty, & Q = Q_o \end{cases}$$
(2.34a)

$$\int_{\Omega} \delta(Q - Q_o) \, d\,\Omega_Q = \int_{\Omega^*} \delta(Q - Q_o) \, d\,\Omega_Q = 1,$$

$$Q_o(x_o, y_o) \in \Omega^* \subseteq \Omega$$
(2.34b)

In accordance to the one-dimensional delta function $\delta(x - x_o)$, the two-dimensional $\delta(Q - Q_o)$ may be defined as the limit of a set of functions. More specifically,

$$\delta(Q-Q_o) = \lim_{k \to \infty} f_k(r), \qquad Q(x,y), Q_o(x_o, y_o) \in \Omega$$
(2.35)

where

$$r = \sqrt{(x - x_o)^2 + (y - x_o)^2}$$

For example the limit of the function

$$f_k(r) = \begin{cases} k^2/\pi, & r < \frac{1}{k} \\ 0, & r \ge \frac{1}{k} \end{cases}$$
(2.36a)

or

$$f_k(r) = \frac{ke^{-kr^2}}{\pi}$$
 (2.36b)

for $k \to \infty$ is a two-dimensional delta function. We can also write $\delta(Q - Q_o)$ as the product of two one-dimensional delta functions as follows

$$\delta(Q - Q_o) = \delta(x - x_o) \,\delta(y - y_o) \tag{2.37}$$

Consider now the transformation

$$\begin{array}{c} x = u(\xi, \eta) \\ y = v(\xi, \eta) \end{array} \right\}$$

$$(2.38)$$
where u and v are assumed to be single-valued continuous differentiable functions of their arguments. The transformation of the function $\delta(Q - Q_o)$ from the Cartesian coordinates x and y to the curvilinear coordinates ξ and η can be performed in the following way.

It is supposed that under this transformation the point $x = x_o$, $y = y_o$ is mapped onto the point $\xi = \xi_o$, $\eta = \eta_o$. By changing coordinates, the integral of Eq. (2.33)

$$\int_{\Omega} \delta(x-x_o) \, \delta(y-y_o) \, h(x,y) \, dx \, dy = h(x_o,y_o)$$

becomes

$$\int_{\Omega} \delta[u(\xi,\eta) - x_o] \, \delta[v(\xi,\eta) - y_o] \, h(u,v) \, |J| \, d\xi \, d\eta = h(x_o, y_o) \tag{2.39}$$

where

$$J = \begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{vmatrix}$$

is the Jacobian of the transformation.

Equation (2.39) states that the symbolic function

$$\delta[u(\xi,\eta)-x_o] \, \delta[v(\xi,\eta)-y_o] \, |J|$$

assigns the value of the function h(x,y) at the point where $u = x_o$, $v = y_o$, that is at the point where $\xi = \xi_o$, $\eta = \eta_o$. Consequently, we may write

$$\delta[u(\xi,\eta)-x_o] \, \delta[v(\xi,\eta)-y_o] \, |J| = \delta(\xi-\xi_o) \, \delta(\eta-\eta_o)$$

which may also be set in the form

$$\delta(x - x_o) \,\delta(y - y_o) = \frac{\delta(\xi - \xi_o) \,\delta(\eta - \eta_o)}{|J|} \tag{2.40}$$

provided that $|J| \neq 0$, a condition which implies that the transformation (2.38) is non-singular (invertible).

We close this section by stating a property of the derivatives of the delta function.

(i) For the m^{th} order derivative of the one-dimensional delta function the following equation is valid

$$\int_{a}^{b} h(x) \frac{d^{m} \delta(x - x_{o})}{dx^{m}} dx = (-1)^{m} \frac{d^{m} h(x_{o})}{dx^{m}}, \qquad (a < x_{o} < b)$$
(2.41)

(ii) For the $(m + n)^{th}$ order derivative of the two-dimensional delta function, the following equation is valid

$$\int_{\Omega} h(Q) \frac{\partial^{m+n} \delta(Q-Q_o)}{\partial x^m \partial y^n} d\Omega_Q = (-1)^{m+n} \frac{\partial^{m+n} h(Q_o)}{\partial x^m \partial y^n}$$
(2.42)

where $Q_o(x_o, y_o), Q(x, y) \in \Omega$.

2.7 References

This chapter has been designed to give a brief elementary description of the basic mathematical tools that will be employed throughout this book in developing the BEM formulations. The reader may look for more details about the Gauss-Green theorem and the Gauss divergence theorem in most books on calculus as well as in many books of engineering mathematics or mathematical physics, such as Smirnow [1], Hildebrand [2], Kreyszig [3], Sommerfeld [4]. For a detailed discussion about the Dirac delta function, we refer to the book of Greenberg [5] or to the more advanced books of Roach [6] and Duff and Naylor [7].

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- [2] Hildebrand, F.B., 1962. *Advanced Calculus for Applications*, Prentice Hall, Englewood Cliffs, New Jersey.
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Problems

2.1. Convert the domain integral

$$\int_\Omega f\,d\Omega$$

to boundary line integral on Γ , when the function f is

(i)	f = x	(ii)	f = y
(iii)	f = xy	(iv)	$f = x^2$
(v)	$f=y^2$	(vi)	$f = x^2 + y^2$
(vii)	$f = \cos x$		

2.2. Derive the operator ∇^2 in polar coordinates (r, θ) and convert the following domain integral to a boundary line integral on Γ

$$\int_{\Omega} \ell n r \ d\Omega, \qquad r = \sqrt{x^2 + y^2}$$

2.3. Evaluate the integrals

(i)
$$\int_{a}^{b} \delta(x - x_{o}) dx, \quad a < x_{o} < b$$

(ii)
$$\int_{a}^{b} \delta(kx) f(x) dx, \quad a < 0 < b$$

(iii)
$$\int_{a}^{b} \delta(-x) dx, \quad a < 0 < b$$

(iv)
$$\int_{a}^{b} \delta^{(n)}(x) \phi(x) dx, \quad a < 0 < b$$

2.4. Show that

(i)
$$\delta(-x) = \delta(x)$$
, $a < 0 < b$
(ii) $\delta(ax + by) \,\delta(cx + dy) = \frac{\delta(x)\,\delta(y)}{|ad - bc|}$

- **2.5.** Transform the delta function $\delta(P P_o)$ for the points P(x, y) and $P_o(x_o, y_o)$ into polar coordinates r and θ .
- 2.6. Derive Eq. (2.21) along with Eqs. (2.22) through (2.24).

2.7. Show that
$$\int_{\Omega} \nabla^2 u \, d\Omega = \int_{\Gamma} \frac{\partial u}{\partial n} \, ds$$
.

- **2.8.** Given $L(u) = \nabla^2 u + \mathbf{a} \cdot \nabla u + cu$, where $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$ is an arbitrary vector, show that
 - (i) $L^*(u) = \nabla^2 u \nabla \cdot (\mathbf{a}u) + cu$,
 - (ii) $vL(u) uL^*(v) = \nabla \cdot (v\nabla u u\nabla v + \mathbf{a}uv)$, and
 - (iii) derive Green's identity for the operator L(u).

The BEM for Potential Problems in Two Dimensions

3.1 Introduction

In this chapter the boundary element method is developed for the solution of engineering problems described by the potential equation

$$\nabla^2 u = f(x, y) \qquad (x, y \in \Omega) \tag{3.1}$$

This is the governing equation of potential theory, which for f = 0 is known as the Laplace equation, whereas for $f \neq 0$ is known as the Poisson equation. Its solution u = u(x, y) represents the potential produced at a point (x, y) in the domain Ω due to a source f(x, y) distributed over Ω . The potential equation (3.1) describes the response of many physical systems. It appears in steady state flow problems, such as fluid flow, thermal flow, electricity flow, as well as in torsion of prismatic bars, bending of membranes, etc. According to the definitions given in Section 2.5, Eq. (3.1) is of elliptic type, since $\Delta < 0$. Its solution is sought in a closed plane domain Ω having a boundary Γ on which either the function u or its derivative $\partial u/\partial n$ in the direction normal to Γ is prescribed. That is, the solution must satisfy the *boundary conditions* of the problem on the boundary Γ . The *boundary value problems* for the potential equation can be classified as follows:

(i) Dirichlet problem

 $\nabla^2 u = f \quad \text{in } \Omega \tag{3.2a}$

$$u = \overline{u} \quad \text{on} \quad \Gamma \tag{3.2b}$$

(ii) Neumann problem

$$\nabla^2 u = f \quad \text{in } \Omega \tag{3.3a}$$

$$\frac{\partial u}{\partial n} = \overline{u}_n \quad \text{on } \Gamma$$
 (3.3b)

(iii) Mixed problem

$$\nabla^2 u = f \quad \text{in } \Omega \tag{3.4a}$$

$$u = \overline{u}$$
 on Γ_1 (3.4b)

$$\frac{\partial u}{\partial n} = \overline{u}_n \quad \text{on} \quad \Gamma_2$$
(3.4c)

where $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \{\emptyset\}$.

(iv) Robin problem

$$\nabla^2 u = f \quad \text{in } \Omega \tag{3.5a}$$

$$u + k(s)\frac{\partial u}{\partial n} = 0$$
 on Γ (3.5b)

The quantities denoted by \overline{u} , \overline{u}_n and k(s) are known functions defined on the boundary.

All four problems can be expressed through a single formulation as

$$\nabla^2 u = f \quad \text{in } \Omega \tag{3.6a}$$

$$\alpha u + \beta \frac{\partial u}{\partial n} = \gamma \quad \text{on} \quad \Gamma$$
 (3.6b)

where $\alpha = \alpha(s)$, $\beta = \beta(s)$ and $\gamma = \gamma(s)$ are known functions defined on the boundary Γ . Apparently, each of the foregoing four boundary value problems (Eqs. 3.2-3.5) may be derived from Eq. (3.6) by appropriately specifying the functions α , β and γ .

Two boundary element methods have been developed for solving the previously stated four boundary value problems: the *direct boundary element method* and the *indirect boundary element method*. In this book, we present only the direct boundary element method.

3.2 Fundamental solution

Let us consider a point source placed at point P(x,y) of the xy-plane. Its density at $Q(\xi,\eta)$ may be expressed mathematically by the delta function as

$$f(Q) = \delta(Q - P) \tag{3.7}$$

and the potential v = v(Q, P) produced at point Q satisfies the equation

$$\nabla^2 v = \delta(Q - P) \tag{3.8}$$

A singular particular solution of Eq. (3.8) is called the *fundamental solution* of the potential equation (3.1). It is determined by writing Eq. (3.8) in polar coordinates with origin at the point source P. Since this solution is axisymmetric with respect to the source, it is independent of the polar angle θ , and thus Eq. (3.8) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \delta(Q-P) \tag{3.9}$$

where

$$r = |Q - P| = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$
(3.10)

The right-hand side of Eq. (3.8) vanishes at all points of the plane, except at the origin r = 0, where it has an infinite value. Apart from point r = 0, Eq. (3.9) is written as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0$$

which gives after integrating twice

$$v = A\ell nr + B$$

where A and B are arbitrary constants. Since we look for a particular solution, we may set B = 0. The other constant, A, may be determined in the following way.



Figure 3.1 Circular domain Ω of radius ρ with a source P at its center.

Due to the axisymmetric nature of the problem (see Fig. 3.1), it is

$$\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = A \frac{1}{r}$$
 and $ds = r d\theta$ (3.11)

Application of Green's identity (2.16) for u = 1 and $v = A \ln r$, yields

$$-\int_{\Omega} \nabla^2 v \ d\Omega = -\int_{\Gamma} \frac{\partial v}{\partial n} \ ds$$

where Ω is the circle with center at point P and radius ρ . Using then Eqs. (3.8) and (3.11), and noting that for points on the boundary Γ it is $r = \rho$, the above relation is written as

$$-\int_{\Omega} \delta(Q-P) \, d\Omega = -\int_{0}^{2\pi} A \, \frac{1}{\rho} \, \rho \, d\theta$$

or by applying Eq. (2.34b), it takes the form

$$1 = 2\pi A$$

from which we obtain

$$A = \frac{1}{2\pi} \tag{3.12}$$

Hence, the fundamental solution becomes

$$v = \frac{1}{2\pi} \ell n r \tag{3.13}$$

From Eq. (3.10) it is evident that the fundamental solution does not change value, when points P and Q interchange their role. This means that v is symmetric with respect to these points, namely

$$v(Q,P) = v(P,Q) \tag{3.14}$$

The fundamental solution (3.13) is also known in the literature as the *free space Green's function*.

3.3 The direct BEM for the Laplace equation

In this section, we derive the solution of the Laplace equation

$$\nabla^2 u = 0 \quad \text{in } \Omega \tag{3.15}$$

with mixed boundary conditions (see Fig. 3.2)

$$u = \overline{u}$$
 on Γ_1 (3.16a)

$$\frac{\partial u}{\partial n} = \overline{u}_n \quad \text{on} \quad \Gamma_2$$
(3.16b)

where $\Gamma_1 \cup \Gamma_2 = \Gamma$. In the literature, condition (3.16a) is referred to as *essential* or *kinematic*, while (3.16b) as *natural*. Instead of the boundary conditions (3.16), we can use the general condition (3.6b), but at the present time, we avoid it for the sake of simplicity.

Applying Green's identity (2.16), for the functions u and v that satisfy Eqs. (3.15) and (3.8), respectively, and assuming that the source lies at point P, we obtain

$$-\int_{\Omega} u(Q) \,\delta(Q-P) \,d\Omega_Q = \int_{\Gamma} \left[v(q,P) \,\frac{\partial u(q)}{\partial n_q} - u(q) \,\frac{\partial v(q,P)}{\partial n_q} \right] ds_q \tag{3.17}$$

where $P, Q \in \Omega$ and $q \in \Gamma$.

In the previous equations and hereafter, points inside the domain Ω are denoted by upper case letters, e.g. P, Q, while points on the boundary Γ are denoted by lower case letters, e.g. p, q. The subscripts of the differentials, e.g. $d\Omega_Q$, ds_q , and the derivatives, e.g. $\partial()/\partial n_q$, denote the points that vary during integration or differentiation, respectively.



Figure 3.2 Domain Ω with mixed boundary conditions.

By virtue of Eqs. (2.33) and (3.14), Eq. (3.17) is written as

$$u(P) = -\int_{\Gamma} \left[v(P,q) \frac{\partial u(q)}{\partial n_q} - u(q) \frac{\partial v(P,q)}{\partial n_q} \right] ds_q$$
(3.18)

The functions v and $\partial v/\partial n$ in the foregoing equation are both known quantities. These are the fundamental solution of the Laplace equation and its normal derivative at point q of the boundary, which are given as

$$v = \frac{1}{2\pi} \ell n r \tag{3.19}$$

$$\frac{\partial v}{\partial n} = \frac{1}{2\pi} \frac{\cos \phi}{r} \tag{3.20}$$

where r = |q - P| and $\phi = angle(\mathbf{r}, \mathbf{n})$ (see Appendix A).

The expression (3.18) is the solution of the differential equation (3.15) at any point P inside the domain Ω (*not* on the boundary Γ) in terms of the boundary values of u and its normal derivative $\partial u/\partial n$. The relation (3.18) is called the *integral representation of the solution* for the Laplace equation. It is apparent from the boundary conditions (3.16a) and (3.16b), that only one of the quantities u or $\partial u/\partial n$ is prescribed at a point $q(\xi, \eta)$ on the boundary. Consequently, it is not yet possible to determine the solution from the integral representation (3.18). For this purpose, we are going to evaluate the boundary quantity which is not prescribed by the boundary conditions (either u or $\partial u/\partial n$), by deriving the integral representation of u for points $P \equiv p$ lying on the boundary Γ .



Figure 3.3 Geometric definitions related to a corner point P of a non-smooth boundary.

We study the general case where the boundary is not smooth and P is a corner point (see Fig. 3.3). We consider the domain Ω^* which results from Ω after subtracting a small circular section with center P, radius ε and confined by the arcs PA and PB. The circular arc AB is denoted by Γ_{ε} and the sum of the arcs APand PB by ℓ . The outward normal to Γ_{ε} coincides with the radius ε and is directed towards the center P. The angle between the tangents of the boundary at point P is denoted by α . Obviously, it is

$$\begin{split} & \lim_{\varepsilon \to 0} (heta_1 - heta_2) = lpha \ & \lim_{\varepsilon \to 0} \Gamma_{\varepsilon} = 0 \ & \lim_{\varepsilon \to 0} (\Gamma - \ell) = \Gamma \end{split}$$

Next we apply Green's identity (2.16) in the domain Ω^* for the functions u and v satisfying Eqs. (3.15) and (3.8), respectively. Since point P lies outside the domain Ω^* , where it is $\delta(Q - P) = 0$, it follows that

$$\int_{\Omega^*} u\,\delta(Q-P)\,d\Omega = 0$$

and consequently Green's identity gives

$$0 = \int_{\Gamma-\ell} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds + \int_{\Gamma_{\ell}} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$
(3.21)

We will examine next the behavior of the integrals in the above equation when $\varepsilon \rightarrow 0$. Apparently, the first integral becomes

$$\lim_{\varepsilon \to 0} \int_{\Gamma - \ell} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$
(3.22)

while, the second one is written as (see Appendix A)

$$\int_{\Gamma_{\varepsilon}} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \int_{\Gamma_{\varepsilon}} \frac{1}{2\pi} \frac{\partial u}{\partial n} \ln r \, ds - \int_{\Gamma_{\varepsilon}} \frac{1}{2\pi} \, u \, \frac{\cos \phi}{r} \, ds$$
$$= I_1 + I_2 \tag{3.23}$$

For the circular arc Γ_{ε} , it is $r = \varepsilon$ and $\phi = \pi$. Moreover, $ds = \varepsilon (-d\theta)$, because the angle θ is positive in the counter-clockwise sense, which is opposite to that of increasing s. Therefore, the first of the resulting two integrals in Eq. (3.23), takes the form

$$I_{1} = \int_{\Gamma_{\varepsilon}} \frac{1}{2\pi} \frac{\partial u}{\partial n} \, \ell nr \, ds = \int_{\theta_{1}}^{\theta_{2}} \frac{1}{2\pi} \frac{\partial u}{\partial n} \, \varepsilon \, \ell n\varepsilon \, d(-\theta) \tag{3.24}$$

According to the mean value theorem of integral calculus, the value of an integral is equal to the value of its integrand at some point O within the integration interval multiplied by the length of that interval. Hence,

$$I_{1} = \frac{1}{2\pi} \left[\frac{\partial u}{\partial n} \right]_{O} \varepsilon \, \ell n \varepsilon \, (\theta_{1} - \theta_{2})$$

When $\varepsilon \to 0$, the point *O* of the arc approaches point *P*. Of course, in this case the derivative $[\partial u/\partial n]_P$, though not defined, it is bounded. Nevertheless, it is

$$\lim_{\varepsilon \to 0} \left(\varepsilon \, \ell n \varepsilon \right) = 0$$

which implies that

$$\lim_{\varepsilon \to 0} I_1 = 0 \tag{3.25}$$

In a similar way, the second integral in the right-hand side of Eq. (3.23) may be written as

$$I_2 = -\int_{\Gamma_{\epsilon}} \frac{1}{2\pi} u \, \frac{\cos \phi}{r} \, ds = -\int_{\theta_1}^{\theta_2} \frac{1}{2\pi} u \frac{-1}{\varepsilon} \varepsilon \, d(-\theta)$$

or by applying the mean value theorem

$$I_2 = -\frac{1}{2\pi} u_0 \left(\theta_2 - \theta_1\right) = \frac{\theta_1 - \theta_2}{2\pi} u_0$$

and finally

$$\lim_{\varepsilon \to 0} I_2 = \frac{\alpha}{2\pi} u(P) \tag{3.26}$$

By virtue of Eqs. (3.25) and (3.26), Eq. (3.23) yields

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \left[v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] ds = \frac{\alpha}{2\pi} u(P)$$
(3.27)

Incorporating now the findings of Eqs. (3.22) and (3.27) into Eq. (3.21), the latter gives for $\varepsilon \to 0$

$$\frac{\alpha}{2\pi}u(p) = -\int_{\Gamma} \left[v(p,q)\frac{\partial u(q)}{\partial n_q} - u(q)\frac{\partial v(p,q)}{\partial n_q} \right] ds_q$$
(3.28)

The last expression is the integral representation of the solution for the Laplace equation (3.15) at points $p \in \Gamma$, where the boundary is not smooth. For points p, where the boundary is smooth, it is $\alpha = \pi$ and thus, Eq. (3.28) becomes

$$\frac{1}{2}u(p) = -\int_{\Gamma} \left[v(p,q)\frac{\partial u(q)}{\partial n_q} - u(q)\frac{\partial v(p,q)}{\partial n_q} \right] ds_q$$
(3.29)

A comparison between Eqs. (3.18) and (3.28) reveals that the function u is discontinuous when the point $P \in \Omega$ approaches point $p \in \Gamma$. It exhibits a jump equal to

 $(1 - \alpha/2\pi)$ for corner points (Eq. 3.28), or 1/2 for points on smooth parts of the boundary Γ (Eq. 3.29). When the point *P* is located outside the domain Ω , Green's identity (2.16) gives

$$0 = -\int_{\Gamma} \left[v(P,q) \frac{\partial u(q)}{\partial n_q} - u(q) \frac{\partial v(P,q)}{\partial n_q} \right] ds_q$$
(3.30)

Equations (3.18), (3.29) and (3.30) can be combined in a single general equation as

$$\varepsilon(P) \ u(P) = -\int_{\Gamma} \left[v(P,q) \frac{\partial u(q)}{\partial n_q} - u(q) \frac{\partial v(P,q)}{\partial n_q} \right] ds_q \tag{3.31}$$

where $\varepsilon(P)$ is a coefficient which depends on the position of point P and is defined as

$$\varepsilon(P) = \begin{cases} 1 & \text{for } P \text{ inside } \Omega \\ \frac{1}{2} & \text{for } P \equiv p \text{ on the boundary } \Gamma \\ 0 & \text{for } P \text{ outside } \Omega \end{cases}$$

Equation (3.29) constitutes a compatibility relation between the boundary values of u and $\partial u/\partial n$, meaning that only one of the quantities u and $\partial u/\partial n$ can be prescribed at each point of the boundary. At the same time, Eq. (3.29) can be viewed as an integral equation on the boundary Γ , that is a *boundary integral equation*, with unknown the quantity which is not prescribed by the boundary condition.

In the sequel, we assume a smooth boundary Γ . Thus, for the Dirichlet problem $(u = \overline{u} \text{ on } \Gamma)$, Eq. (3.29) is written as

$$\frac{1}{2}\overline{u} = -\int_{\Gamma} \left(v \frac{\partial u}{\partial n} - \overline{u} \frac{\partial v}{\partial n} \right) ds$$
(3.32a)

in which the only unknown is the function $\partial u/\partial n$ on Γ . For the Neumann problem ($\partial u/\partial n = \overline{u}_n$), Eq. (3.29) becomes

$$\frac{1}{2}u = -\int_{\Gamma} \left(v \,\overline{u}_n - u \,\frac{\partial v}{\partial n} \right) ds \tag{3.32b}$$

with only unknown the function u on Γ . For problems with mixed boundary conditions, Eq. (3.29) is treated as two separate equations (see Eqs. 3.16), namely

$$\frac{1}{2}\overline{u} = -\int_{\Gamma} \left(v \frac{\partial u}{\partial n} - \overline{u} \frac{\partial v}{\partial n} \right) ds \quad \text{on} \quad \Gamma_1$$
(3.33a)

$$\frac{1}{2}u = -\int_{\Gamma} \left(v \,\overline{u}_n - u \frac{\partial v}{\partial n} \right) ds \quad \text{on} \quad \Gamma_2$$
(3.33b)

3.4 The direct BEM for the Poisson equation

In this case, we seek the solution of the boundary value problem which is governed by the Poisson equation

$$\nabla^2 u = f \quad \text{in } \Omega \tag{3.34}$$

and has mixed boundary conditions

$$u = \overline{u}$$
 on Γ_1 (3.35a)

$$\frac{\partial u}{\partial n} = \overline{u}_n \quad \text{on} \quad \Gamma_2$$
 (3.35b)

The solution can be obtained in two different ways, which are presented in the following sections.

3.4.1 Application of Green's identity

The integral representation of the solution is obtained by applying Green's identity (2.16) for the functions u and v that satisfy Eqs. (3.34) and (3.8), respectively,

$$\varepsilon(P) \ u = \int_{\Omega} v f \, d\Omega - \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \tag{3.36}$$

The corresponding boundary integral equation for smooth boundary is

$$\frac{1}{2}u = \int_{\Omega} vf \, d\Omega - \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \tag{3.37}$$

3.4.2 Transformation of the Poisson equation to the Laplace equation

The solution of Eq. (3.34) can be obtained as a sum of two solutions

$$u = u_0 + u_1 \tag{3.38}$$

where u_0 is the solution of the homogeneous equation (Laplace equation) with boundary conditions $u_0 = \overline{u} - u_1$ on Γ_1 and $\partial u_0 / \partial n = \overline{u}_n - \partial u_1 / \partial n$ on Γ_2 , and u_1 is a particular solution of the non-homogeneous equation.

(i) Particular solution u_1

The particular solution of the non-homogeneous equation is any function u_1 that satisfies only the governing equation

$$\nabla^2 u_1 = f \tag{3.39}$$

independently of boundary conditions. We will show next that

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$$u_1 = \int_{\Omega} v f \, d\Omega \tag{3.40}$$

where v is the fundamental solution. Using the definition (2.33) of the Dirac delta function, Eq. (3.39) becomes

$$abla^2 u_1(P) = f(P)$$

$$= \int_{\Omega} \delta(Q - P) f(Q) \, d\Omega_Q$$

and incorporating Eq. (3.8),

$$\nabla^2 u_1(P) = \int_{\Omega} \nabla^2 v(Q, P) f(Q) \, d\Omega_Q = \nabla^2 \left[\int_{\Omega} v(Q, P) f(Q) \, d\Omega_Q \right]$$

or

$$\nabla^2 \Big[u_1(P) - \int_{\Omega} v(Q, P) f(Q) \, d\Omega_Q \Big] = 0$$

which is obviously satisfied by the particular solution

$$u_1(P) = \int_{\Omega} v(Q, P) f(Q) \, d\Omega_Q$$

It should be noted that the differentiations indicated by the ∇^2 operator are performed with respect to the coordinates of point P. Moreover, v is continuous with respect to this point. Based on these, the ∇^2 operator was moved outside the integral in the above equations.

Another method to establish a particular solution is by transforming Eq. (3.39) into the complex domain. The transformation is defined as

$$z = x + iy$$
, $\overline{z} = x - iy$ $(i = \sqrt{-1})$ (3.41a)

and its inverse is

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}$$
 (3.41b)

We can readily show that Eq. (3.39) is transformed to

$$4\frac{\partial^2 u_1}{\partial z \,\partial \overline{z}} = f(z,\overline{z}) \tag{3.42}$$

which, after two successive integrations, yields the particular solution $u_1(z, \overline{z})$. It is worth mentioning that any arbitrary integration functions can be omitted, because we look only for a particular solution. Subsequently, the transformation (3.41a) produces the particular solution $u_1(x, y)$ in the physical space.

Example 3.1

Determine a particular solution $u_1(x, y)$ of the Poisson equation (3.34) when

$$f = x^2 + y^2 \tag{a}$$

Applying the transformation (3.41b) to the above equation (a), gives the function f in the complex domain,

$$f = z \overline{z}$$
 (b)

Equation (3.42) may then be written as

$$\frac{\partial^2 u_1}{\partial z \, \partial \overline{z}} = \frac{1}{4} \, z \, \overline{z} \tag{c}$$

which, after successive integrations, yields

$$u_1 = \frac{1}{16} z^2 \,\overline{z}^2 \tag{d}$$

Substituting Eq. (3.41b) in the above equation, we find the particular solution in the physical domain

$$u_1 = \frac{1}{16}(x^2 + y^2)^2 \tag{e}$$

(ii) Homogeneous solution u_0

Once the particular solution u_1 has been established, the solution u_0 of the homogeneous equation will be obtained from the boundary value problem

$$\nabla^2 u_0 = 0 \quad \text{in } \Omega \tag{3.43}$$

$$u_0 = \overline{u} - u_1 \quad \text{on} \ \ \Gamma_1 \tag{3.44a}$$

$$\frac{\partial u_0}{\partial n} = \overline{u}_n - \frac{\partial u_1}{\partial n}$$
 on Γ_2 (3.44b)

This problem can be solved using the procedure presented in Section 3.3.

3.5 Transformation of the domain integrals to boundary integrals

In the process of solving the Poisson equation by BEM, domain integrals appear in the integral representation of the solution (3.36). These integrals are of the form

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$$\int_{\Omega} v f \, d\Omega \tag{3.45}$$

Although the integrand vf is known, the fact that domain integrals need to be evaluated spoils the pure boundary character of the method, thus weakening the advantages of BEM over domain methods. However, it is possible to overcome this drawback by converting the domain integral (3.45) to a boundary line integral on Γ , i.e., the boundary of the domain Ω . Next, two different approaches are presented for the conversion of integral (3.45).

(i) The function f is polynomial of x and y.

Let us assume that the function f is a first degree polynomial with respect to x and y,

$$f = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{3.46}$$

where α_0 , α_1 and α_2 are known constants. This function, apparently, satisfies the Laplace equation,

$$\nabla^2 f = 0 \tag{3.47}$$

As it will be shown later, we may determine a function U which satisfies the Poisson equation

$$\nabla^2 U = v \tag{3.48}$$

and apply Green's identity (2.16) for the functions f and U,

$$\int_{\Omega} \left(f \, \nabla^2 U - U \, \nabla^2 f \right) d\,\Omega = \int_{\Gamma} \left(f \, \frac{\partial U}{\partial n} - U \, \frac{\partial f}{\partial n} \right) ds$$

By virtue of Eqs. (3.47) and (3.48), the above equation yields

$$\int_{\Omega} vf \, d\Omega = \int_{\Gamma} \left(f \frac{\partial U}{\partial n} - U \frac{\partial f}{\partial n} \right) ds \tag{3.49}$$

If the function f is defined in a subdomain $\Omega^* \subset \Omega$ with boundary Γ^* (see Fig. 3.4), then Eq. (3.49) becomes

$$\int_{\Omega} vf \, d\Omega = \int_{\Omega} vf \, d\Omega = \int_{\Gamma} \left(f \frac{\partial U}{\partial n} - U \frac{\partial f}{\partial n} \right) ds \tag{3.50}$$

The function U = U(r) is established as a particular solution of Eq. (3.48). For this purpose, we write the Laplace operator in polar coordinates,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) = \frac{1}{2\pi}\,\ell nr$$



Figure 3.4 Subdomain $\Omega^* \subset \Omega$ with boundary Γ^* .

Integrating twice the above equation, we find the particular solution

$$U = \frac{1}{8\pi} r^2 (\ell n r - 1)$$
(3.51)

(ii) The function f is arbitrary.

In this case f is an arbitrary function defined in $\Omega^* \subset \Omega$. We first establish another function F which satisfies the following Poisson equation

$$\nabla^2 F = f \tag{3.52}$$

This function F is determined as a particular solution of Eq. (3.52) using the procedure presented in Section 3.4.2. The Green identity (2.16) is then applied for the functions v and F in the domain Ω^* ,

$$\int_{\Omega^*} \left(v \, \nabla^2 F - F \, \nabla^2 v \right) d\Omega = \int_{\Gamma^*} \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds$$

which becomes

$$\int_{\Omega^{\star}} vf \, d\Omega = \int_{\Omega^{\star}} F \, \delta(Q - P) \, d\Omega_Q + \int_{\Gamma^{\star}} \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds \tag{3.53}$$

We distinguish two cases:

(a) $\Omega^* \equiv \Omega$. The point P is always inside Ω and Eq. (3.53) yields

$$\int_{\Omega} vf \, d\Omega = F + \int_{\Gamma} \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds \tag{3.54}$$

(b) Ω^{*} ⊂ Ω. The point P may be inside Ω^{*}, on Γ^{*}, or outside Ω^{*}. In this case Eq. (3.53) gives

$$\int_{\Omega^{\star}} vf \, d\Omega = \varepsilon(P) F + \int_{\Gamma^{\star}} \left(v \frac{\partial F}{\partial n} - F \frac{\partial v}{\partial n} \right) ds \tag{3.55}$$

where $\varepsilon(P) = 1, \frac{1}{2}, 0$ depending on whether the point P is inside Ω^* , on Γ^* or outside Ω^* , respectively.

3.6 The BEM for potential problems in anisotropic bodies

In this section, we develop the BEM solution for the following boundary value problem

$$k_{xx}\frac{\partial^2 u}{\partial x^2} + 2k_{xy}\frac{\partial^2 u}{\partial x \partial y} + k_{yy}\frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{in } \Omega$$
(3.56a)

$$u = \overline{u} \quad \text{on} \ \Gamma_1$$
 (3.56b)

$$\nabla u \cdot \mathbf{m} = \overline{q}_n \quad \text{on} \quad \Gamma_2 \tag{3.56c}$$

where

$$\mathbf{m} = (k_{xx} n_x + k_{xy} n_y) \mathbf{i} + (k_{xy} n_x + k_{yy} n_y) \mathbf{j}$$

is a vector in the direction of the connormal to the boundary and k_{xx} , k_{xy} , k_{yy} are constants satisfying the ellipticity condition $k_{xy}^2 - k_{xx}k_{yy} < 0$. Obviously, when $k_{xx} = k_{yy} = 1$ and $k_{xy} = 0$, it is $\mathbf{m} = \mathbf{n}$ and the boundary value problem (3.56) reduces to the mixed boundary value problem for the Poisson equation described by Eqs. (3.34) and (3.35). Equation (3.56a) describes potential problems in anisotropic bodies (see for example Section 6.5).

3.6.1 Integral representation of the solution

Green's identity for the differential operator of Eq. (3.56a) can be obtained from Eq. (2.21) by setting $A = k_{xx}$, $B = k_{xy}$, $C = k_{yy}$ and D = E = F = 0. Thus, one readily arrives at the following equation

$$\int_{\Omega} \left[v L(u) - u L(v) \right] d\Omega = \int_{\Gamma} \left(v \nabla u \cdot \mathbf{m} - u \nabla v \cdot \mathbf{m} \right) ds$$
(3.57)

where

$$L(\) = k_{xx}\frac{\partial^2}{\partial x^2} + 2k_{xy}\frac{\partial^2}{\partial x\partial y} + k_{yy}\frac{\partial^2}{\partial y^2}$$
(3.58)

The boundary condition (3.56c) may also be written, after rearranging its terms, as

$$\nabla u \cdot \mathbf{m} = \mathbf{q} \cdot \mathbf{n} = q_n \tag{3.59}$$

where

$$\mathbf{q} = q_x \,\mathbf{i} + q_y \,\mathbf{j} = \left(k_{xx} \frac{\partial u}{\partial x} + k_{xy} \frac{\partial u}{\partial y}\right) \mathbf{i} + \left(k_{xy} \frac{\partial u}{\partial x} + k_{yy} \frac{\partial u}{\partial y}\right) \mathbf{j}$$
(3.60)

is a vector representing the *flux* of u, which is the flow of the potential u through the boundary Γ per unit length of the boundary, and q_n its component (projection) in the direction of the normal to the boundary.

The resulting integral representation of the solution is obtained as

$$u(P) = \int_{\Omega} vf \, d\Omega - \int_{\Gamma} \left(v \, \nabla u \cdot \mathbf{m} - u \, \nabla v \cdot \mathbf{m} \right) ds \,, \quad P \in \Omega \tag{3.61}$$

where v is the fundamental solution of Eq. (3.56a), namely, a singular particular solution of the equation

$$L(v) = \delta(Q - P) \tag{3.62}$$

3.6.2 Fundamental solution

The fundamental solution of Eq. (3.56a) is established by transforming Eq. (3.62) in its canonical form. For this purpose we consider the transformation

$$\left. \begin{array}{c} \overline{x} = y - a \, x \\ \overline{y} = -b \, x \end{array} \right\} \tag{3.63a}$$

and its inverse

$$x = -\frac{1}{b} \overline{y}$$

$$y = \overline{x} - \frac{a}{b} \overline{y}$$
(3.63b)

where

$$a = \frac{k_{xy}}{k_{xx}}$$
 and $b = \frac{\sqrt{|D|}}{k_{xx}}$ (3.64)

$$|D| = \begin{vmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{vmatrix} = k_{xx} k_{yy} - k_{xy}^2$$
(3.65)

The derivatives involved in the operator (3.58) become under the transformation (3.63a)

$$\begin{split} \frac{\partial v}{\partial x} &= -\left(a\frac{\partial v}{\partial \overline{x}} + b\frac{\partial v}{\partial \overline{y}}\right) \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial \overline{x}} \\ \frac{\partial^2 v}{\partial x^2} &= a^2\frac{\partial^2 v}{\partial \overline{x}^2} + 2ab\frac{\partial^2 v}{\partial \overline{x}\partial \overline{y}} + b^2\frac{\partial^2 v}{\partial \overline{y}^2} \\ \frac{\partial^2 v}{\partial x\partial y} &= -\left(a\frac{\partial^2 v}{\partial \overline{x}^2} + b\frac{\partial^2 v}{\partial \overline{x}\partial \overline{y}}\right) \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 v}{\partial \overline{x}^2} \end{split}$$

and, finally, the operator L(v) takes the following concise form

$$L(v) = \frac{|D|}{k_{xx}} \left(\frac{\partial^2 v}{\partial \overline{x}^2} + \frac{\partial^2 v}{\partial \overline{y}^2} \right)$$
(3.66)

The function $\delta(Q - P)$ is transformed into the $\overline{x}\overline{y}$ -plane using Eq. (2.40),

$$\delta(Q-P) = \frac{\delta(\bar{Q}-\bar{P})}{|J|}$$

where the coordinates of the points are $Q(\xi,\eta)$, P(x,y), $\overline{Q}(\overline{\xi},\overline{\eta})$, $\overline{P}(\overline{x},\overline{y})$ and the Jacobian of the transformation (3.63a) becomes

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \overline{x}} & \frac{\partial y}{\partial \overline{x}} \\ \frac{\partial x}{\partial \overline{y}} & \frac{\partial y}{\partial \overline{y}} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{1}{b} & -\frac{a}{b} \end{vmatrix} = \frac{1}{b}$$

or

$$|J| = \frac{k_{xx}}{\sqrt{|D|}} \tag{3.67}$$

Consequently, the Dirac function is being transformed as

$$\delta(Q-P) = \frac{\sqrt{|D|}}{k_{xx}} \,\delta\big(\bar{Q}-\bar{P}\big) \tag{3.68}$$

Incorporating Eqs. (3.66) and (3.68) into Eq. (3.62), the latter becomes

$$\nabla^2 v = \frac{1}{\sqrt{|D|}} \,\delta(\bar{Q} - \bar{P}) \tag{3.69}$$

where the Laplace operator is expressed in terms of the \overline{x} and \overline{y} coordinates. A comparison between Eqs. (3.8) and (3.69) reveals that, according to Eq. (3.13), the fundamental solution of Eq. (3.69) in the $(\overline{x}, \overline{y})$ coordinate system is

$$v = \frac{1}{2\pi\sqrt{|D|}} \,\ell nr \tag{3.70}$$

where

$$r = \sqrt{\left(\overline{\xi} - \overline{x}\,
ight)^2 + \left(\,\overline{\eta} - \overline{y}\,
ight)^2}$$

or back in the (x, y) system

$$r = \sqrt{\left(a^{2} + b^{2}\right)\left(\xi - x\right)^{2} - 2a\left(\xi - x\right)\left(\eta - y\right) + \left(\eta - y\right)^{2}}$$

which by virtue of Eq. (3.64) becomes

$$r = \frac{\sqrt{k_{yy} \left(\xi - x\right)^2 - 2k_{xy} \left(\xi - x\right) \left(\eta - y\right) + k_{xx} \left(\eta - y\right)^2}}{\sqrt{k_{xx}}}$$
(3.71)

The fundamental solution is not altered, if a constant is added to it. Thus, we can write

$$v = \frac{1}{2\pi\sqrt{|D|}} \,\ell nr + \frac{1}{2\pi\sqrt{|D|}} \,\ell n \left(\sqrt{\frac{k_{xx}}{|D|}}\right)$$

or

$$v = \frac{1}{2\pi\sqrt{|D|}} \ell nr$$
(3.72)

where now

$$r = \sqrt{\frac{k_{yy} \left(\xi - x\right)^2 - 2 k_{xy} \left(\xi - x\right) \left(\eta - y\right) + k_{xx} \left(\eta - y\right)^2}{|D|}}$$
(3.73)

It should be noted that the quantities $k_{yy}/|D|$, $k_{xy}/|D|$ and $k_{xx}/|D|$ are the elements of the matrix \mathbf{D}^{-1} [17]. If the material is orthotropic, the constants become $k_{xy} = 0$, $|D| = k_{xx} k_{yy}$, and Eq. (3.73) is reduced to

$$r = \sqrt{\frac{(\xi - x)^2}{k_{xx}} + \frac{(\eta - y)^2}{k_{yy}}}$$
(3.74)

For the established fundamental solution (3.72), the term $\nabla v \cdot \mathbf{m}$ appearing in the integral representation (3.61), becomes

$$\nabla v \cdot \mathbf{m} = \frac{1}{2\pi\sqrt{|D|}} \frac{\nabla r \cdot \mathbf{m}}{r}$$
(3.75)

where

$$\nabla r \cdot \mathbf{m} = \frac{\partial r}{\partial \xi} m_x + \frac{\partial r}{\partial \eta} m_y \tag{3.76}$$

in which the components of **m** are $m_x = k_{xx}n_x + k_{xy}n_y$ and $m_y = k_{xy}n_x + k_{yy}n_y$. The derivatives in Eq. (3.76) are obtained by differentiating Eq. (3.73),

$$\frac{\partial r}{\partial \xi} = \frac{1}{|D|} \left(k_{yy} \frac{\xi - x}{r} - k_{xy} \frac{\eta - y}{r} \right)$$
(3.77)

and

$$\frac{\partial r}{\partial \eta} = \frac{1}{|D|} \left(-k_{xy} \frac{\xi - x}{r} + k_{xx} \frac{\eta - y}{r} \right)$$
(3.78)

3.6.3 Boundary integral equation

The boundary integral equation can be obtained from Eq. (3.61) when the point $P \in \Omega$ coincides with a point p on the boundary Γ . In this case, taking into account Eqs. (3.72) and (3.75) and the analysis presented in Section 3.3, for points p where the boundary is smooth, we have

$$\begin{split} & \lim_{P \to p} \int_{\Gamma_{\epsilon}} v \, \nabla u \cdot \mathbf{m} \, \, ds = 0 \\ & \lim_{P \to p} \int_{\Gamma_{\epsilon}} u \, \nabla v \cdot \mathbf{m} \, \, ds = \frac{1}{2} \, u(p) \end{split}$$

Consequently, the boundary integral equation takes the form

$$\frac{1}{2}u(p) = \frac{1}{2\pi\sqrt{|D|}} \int_{\Omega} f \ell nr \, d\Omega$$
$$-\frac{1}{2\pi\sqrt{|D|}} \int_{\Gamma} \left[(\nabla u \cdot \mathbf{m}) \, \ell nr - u \, \frac{\nabla r \cdot \mathbf{m}}{r} \right] ds$$
(3.79)

The kernels in the above equation are identical to those of Eqs. (3.28) or (3.37), except for the distance r, which is given by Eq. (3.73).

Equation (3.79) is used to determine the quantities u and $\nabla u \cdot \mathbf{m}$ on the parts of the boundary where these are not prescribed. Subsequently, the solution at any point $P \in \Omega$ is evaluated from Eq. (3.61).

3.7 References

The method of solving the potential equation by transforming it into a boundary integral equation is not recent but it goes back to the beginning of the last century. In 1900, Fredholm [1] employed it in the potential theory to determine the unknown boundary quantities from the prescribed ones. A detailed presentation of the method can be found in Kellog's book [2] on potential theory. However, the researchers abandoned the boundary integral equation method, as a solution method for potential problems, since it was practically impossible to solve the resulting singular boundary integral equations. The method was mainly limited to prove existence theorems for the solution of the differential equations.

Nevertheless, with the advent of modern computers the method came forth again and started slowly to be used as a computational method in the beginning of the 1960's. Jaswon [3] and Symm [4], presented a solution algorithm for the potential equation; Jaswon and Ponter [5] developed a numerical technique to solve the boundary integral equation for the classical Saint-Venant torsion problem of noncircular bars. They adopted a formulation in terms of the warping function and solved a Neumann problem for the Laplace equation. Numerical results were obtained for bars with various cross-sectional shapes, such as ellipses with or without holes, rectangles, equilateral triangles and circles with notches. Mendelson [6] solved the same problem as a Dirichlet problem for the stress function. Later, Katsikadelis and Sapountzakis [7] formulated the boundary integral equations for the torsion problem of composite bars consisting of two or more materials. They developed the numerical technique for the solution of the boundary integral equations and produced numerical results for composite cross-sections of elliptical shape with elliptical or circular inclusions, for hollow box-shaped cross-sections and for a rectangular cross-section with an I-shaped inclusion. Symm [8] solved the problem of conformal mapping for simply connected domains having arbitrary shapes onto the unit circle |w(z)| = 1 of the complex domain. Christiansen [9] presented a complete collection of the integral equations for the Saint-Venant torsion problem.

In the last 20 years, the number of publications on BEM solutions for the potential equations has increased enormously. The reader is advised to look in the *Boundary Elements Communications* [10] or in the recently published *Boundary Elements Reference Database* [11]. For a more detailed study of the boundary integral equation method, we refer to the books by Jaswon and Symm [12], and Gipson [13]. For the application of BEM to the potential problems, we refer to the books by Banerjee and Butterfield [14], Brebbia and Dominguez [15]. Finally, it is worth mentioning the books by Zauderer [16] and Mikhlin [17], which may be useful to readers interested in the theory of partial differential equations of elliptic type.

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Problems

3.1. Determine a partial solution of the equation

$$\nabla^2 u = x y$$

3.2. Evaluate the following integral over a circle Ω having radius R and being centered at point $P(x_o, y_o)$

$$\int_\Omega vf\,d\Omega$$

where

$$v = \frac{1}{2\pi} \ell nr$$
, $r = |Q - P| = \sqrt{(x - x_o)^2 + (y - y_o)^2}$

and

$$f(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$$

3.3. Convert the domain integral

$$\int_{\Omega} \frac{\partial v}{\partial x} f \, d\Omega$$

where

$$v = \frac{1}{2\pi} \ell n r$$
, $r = |Q - P| = \sqrt{(x - x_o)^2 + (y - y_o)^2}$

to a line integral on the boundary Γ , for the following two cases

- (i) $f = \alpha_0 + \alpha_1 x + \alpha_2 y$
- (ii) $f = x^2 + y^2$
- **3.4.** Derive the integral representations of the derivatives $\partial u/\partial n_p$ and $\partial u/\partial t_p$, where u is the solution of the Laplace equation and n_p and t_p are the directions of the outward normal and the tangent to the boundary at point $p \in \Gamma$, respectively.
- 3.5. Derive the integral representations of the derivatives ∂u/∂x and ∂u/∂y at points: (i) P ∈ Ω, and (ii) p ∈ Γ.

Chapter **4**

Numerical Implementation of the BEM

4.1 Introduction

This chapter presents the numerical implementation of the BEM for solving the potential problems analyzed in the previous chapter. For realistic engineering applications, an exact solution of the integral equation (3.29) is out of the question. A numerical, however, solution of the same equation is always feasible by employing the BEM.

Let us consider an arbitrary domain Ω with boundary Γ . The quintessence of the BEM is to discretize the boundary into a finite number of segments, not necessarily equal, which are called boundary elements. Two approximations are made over each of these elements. One is about the geometry of the boundary, while the other has to do with the variation of the unknown boundary quantity over the element. The usually employed boundary elements are the *constant element*, the *linear ele*ment and the parabolic or quadratic element. On each element, we distinguish the extreme points or end points and the nodes or nodal points. The latter are the points at which values of the boundary quantities are assigned. In the case of constant elements the boundary segment is approximated by a straight line, which connects its extreme points. The node is placed at the mid-point of the straight line and the boundary quantity is assumed to be constant along the element and equal to its value at the nodal point (Fig. 4.1a). For linear elements, the boundary segment is approximated again by a straight line connecting its end points. The element has two nodes usually placed at the extreme points and the boundary quantity is assumed to vary linearly between the nodal values (Fig. 4.1b). Finally, for parabolic elements, the geometry is approximated by a parabolic arc. The element has three nodes, two of which are placed at the ends and the third somewhere in-between, usually at the mid-point (Fig. 4.1c).



Figure 4.1 Various types of boundary elements.

For linear and parabolic elements, the geometry of the segment is depicted isopara*metrically*, that is the geometry and the boundary quantity are approximated over the element by polynomials of the same degree. For constant elements, the geometry is depicted *superparametrically*, since it is represented with higher-order polynomial than that used to approximate the boundary quantity. Constant elements depict the boundary quantities discontinuously from element to element, in contrast to linear and parabolic elements, which depict them continuously. Although, the inter-element continuity ensures a better approximation of the boundary quantity, it gives rise to complications at corner points or at points where the boundary conditions change type (mixed boundary value problems). These difficulties can be overcome by employing discontinuous linear and parabolic elements, which do not have nodal points at the ends of the element (see Chapter 5). The numerical solution of the integral equation (3.29) will be first presented by using constant boundary elements, because at this stage understanding the numerical implementation of the BEM overrides the need to incorporate more advanced numerical techniques, which improve the accuracy and efficiency of the BEM.

4.2 The BEM with constant boundary elements

The boundary Γ is discretized into N constant elements, which are numbered in the counter-clockwise sense. The values of the boundary quantity u and its normal derivative $\partial u/\partial n$ (denoted also as u_n) are assumed constant over each element and equal to their value at the mid-point of the element.

The discretized form of Eq. (3.29) is expressed for a given point p_i on Γ as

$$\frac{1}{2}u^{i} = -\sum_{j=1}^{N} \int_{\Gamma_{j}} v(p_{i},q) \frac{\partial u(q)}{\partial n_{q}} ds_{q} + \sum_{j=1}^{N} \int_{\Gamma_{j}} u(q) \frac{\partial v(p_{i},q)}{\partial n_{q}} ds_{q}$$
(4.1)

where Γ_j is the segment (straight line) on which the j-th node is located and over which integration is carried out, and p_i is the nodal point of the i-th element. For constant elements, the boundary is smooth at the nodal points, hence $\varepsilon(P) = 1/2$. Moreover, the values of u and $\partial u/\partial n$ are constant on each element, so they can be moved outside the integral. Denoting by u^j and u_n^j the values of u and u_n , respectively, on the j-th element, Eq. (4.1) may be written as

$$-\frac{1}{2}u^{i} + \sum_{j=1}^{N} \left(\int_{\Gamma_{j}} \frac{\partial v}{\partial n} ds \right) u^{j} = \sum_{j=1}^{N} \left(\int_{\Gamma_{j}} v ds \right) u_{n}^{j}$$
(4.2)

The integrals involved in the above equation relate the node p_i , where the fundamental solution is applied, to the node p_j (j = 1, 2, ..., N) (Fig. 4.2). Their values express the contribution of the nodal values u^j and u_n^j to the formation of the value $\frac{1}{2}u^i$. For this reason, they are often referred to as *influence coefficients*. These coefficients are denoted by \hat{H}_{ij} and G_{ij} , which are defined as

$$\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial v(p_i, q)}{\partial n_q} \, ds \quad \text{and} \quad G_{ij} = \int_{\Gamma_j} v(p_i, q) \, ds \tag{4.3}$$

where the point p_i remains constant (reference point), while the point q varies over the j-th element (integration point).



Figure 4.2 Nodal-point location and relative distances for constant element discretization.

Introducing the notation (4.3) into Eq. (4.2), the discrete form of the solution becomes

$$-\frac{1}{2}u^{i} + \sum_{j=1}^{N} \hat{H}_{ij}u^{j} = \sum_{j=1}^{N} G_{ij}u^{j}_{n}$$
(4.4)

Moreover, setting

$$H_{ij} = \hat{H}_{ij} - \frac{1}{2}\delta_{ij} \tag{4.5}$$

where δ_{ij} is the *Kronecker delta*, which is defined as $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for i = j, Eq. (4.4) may further be written as

$$\sum_{j=1}^{N} H_{ij} u^{j} = \sum_{j=1}^{N} G_{ij} u_{n}^{j}$$
(4.6)

Equation (4.6) is applied consecutively for all the nodes p_i (i = 1, 2, ..., N) yielding a system of N linear algebraic equations, which are arranged in matrix form

$$[H]\{u\} = [G]\{u_n\}$$
(4.7)

where [H] and [G] are $N \times N$ square matrices, and $\{u\}$ and $\{u_n\}$ are vectors of dimension N.

Let us assume mixed boundary conditions. In this case, the part Γ_1 of the boundary on which u is described and the part Γ_2 on which u_n is described, are discretized into N_1 and N_2 constant elements, respectively ($\Gamma_1 \cup \Gamma_2 = \Gamma$, $N_1 + N_2 = N$). Hence, Eq. (4.7) again contains N unknowns, that is $N - N_1$ values of u on Γ_2 and $N - N_2$ values of u_n on Γ_1 . These N unknown quantities may be determined from the system of Eqs. (4.7).

Prior to solving the system of the equations, it is necessary to separate the unknown from the known quantities. Equations (4.7) may be written after partitioning of the matrices in the following way

$$\begin{bmatrix} [H_{11}] & [H_{12}] \end{bmatrix} \begin{cases} \{\overline{u}\}_1 \\ \{u\}_2 \end{cases} = \begin{bmatrix} [G_{11}] & [G_{12}] \end{bmatrix} \begin{cases} \{u_n\}_1 \\ \{\overline{u}_n\}_2 \end{cases}$$
(4.8)

where $\{\overline{u}\}_1$ and $\{\overline{u}_n\}_2$ denote the prescribed quantities on Γ_1 and Γ_2 , respectively, while $\{u_n\}_1$ and $\{u\}_2$ denote the corresponding unknown ones. Carrying out the multiplications and moving all the unknowns to the left hand side of the equation, we obtain

$$[A]\{X\} = \{B\}$$
(4.9)

where

$$[A] = \begin{bmatrix} [H_{12}] & -[G_{11}] \end{bmatrix}$$
(4.10a)

$$\{X\} = \begin{cases} \{u\}_2\\ \{u_n\}_1 \end{cases}$$
(4.10b)

$$\{B\} = -[H_{11}] \{\overline{u}\}_{1} + [G_{12}] \{\overline{u}_{n}\}_{2}$$
(4.10c)

[A] being a square matrix with dimensions $N \times N$, and $\{X\}$, $\{B\}$ vectors with dimension N.

The previous rearrangement of matrices is effective when the N_1 points where the values of u are prescribed, thus also the N_2 points where the values of u_n are prescribed, are consecutive. Otherwise, the partitioning of the matrices in Eq. (4.8) should be preceded by an appropriate rearrangement of columns in [H] and [G]. Matrices [A] and $\{B\}$ can also be constructed using another more straightforward procedure, which is based on the observation that matrix [A] will eventually contain all the columns of [H] and [G] that correspond to the unknown boundary values of u and u_n , whereas vector $\{B\}$ will result as the sum of those columns of [H] and [G], which correspond to the known values u and u_n , after they have been multiplied by these values. It should be noted that a change of sign occurs,

when the columns of [G] or [H] are moved to the other side of the equation. The aforementioned procedure is more suitable for writing the computer program.

The solution of the simultaneous equations (4.9) yields the unknown boundary quantities u and u_n . Therefore, knowing all the boundary quantities on Γ , the solution u can be computed at any point P(x, y) in the domain Ω by virtue of Eq. (3.31) for $\epsilon(P) = 1$. Applying the same discretization as in Eq. (4.1), we arrive at the following expression

$$u(P) = \sum_{j=1}^{N} \hat{H}_{ij} u^j - \sum_{j=1}^{N} G_{ij} u_n^j$$
(4.11)

The coefficients G_{ij} and \hat{H}_{ij} are computed again from the integrals (4.3), but in this case the boundary point p_i is replaced in the expressions by the field point P in Ω (see Fig. 4.2).

The partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ can be evaluated at points within Ω by direct differentiation of Eq. (3.31) for $\varepsilon(P) = 1$. Since the fundamental solution and its derivatives are continuous functions of x and y, the differentiation passes under the integral sign giving

$$\frac{\partial u}{\partial x} = -\int_{\Gamma} \left[\frac{\partial v}{\partial x} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial n} \right) \right] ds$$
(4.12)

$$\frac{\partial u}{\partial y} = -\int_{\Gamma} \left[\frac{\partial v}{\partial y} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial n} \right) \right] ds$$
(4.13)

where the derivatives of the fundamental solution (3.13) are obtained as

$$\frac{\partial v}{\partial x} = \frac{1}{2\pi} \frac{1}{r} \frac{\partial r}{\partial x}, \quad \frac{\partial v}{\partial y} = \frac{1}{2\pi} \frac{1}{r} \frac{\partial r}{\partial y}, \quad \frac{\partial v}{\partial n} = \frac{1}{2\pi} \frac{1}{r} \frac{\partial r}{\partial n}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial n}\right) = -\frac{1}{2\pi r^2} \left(\frac{\partial r}{\partial n} \frac{\partial r}{\partial x} - \frac{\partial r}{\partial t} \frac{\partial r}{\partial y}\right)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial v}{\partial n}\right) = -\frac{1}{2\pi r^2} \left(\frac{\partial r}{\partial n} \frac{\partial r}{\partial y} + \frac{\partial r}{\partial t} \frac{\partial r}{\partial x}\right)$$

$$\frac{\partial r}{\partial x} = -\frac{\partial r}{\partial \xi} = -\frac{\xi - x}{r}, \quad \frac{\partial r}{\partial y} = -\frac{\partial r}{\partial \eta} = -\frac{\eta - y}{r}$$

$$\frac{\partial r}{\partial n} = \nabla r \cdot \mathbf{n} = \frac{\partial r}{\partial \xi} n_x + \frac{\partial r}{\partial \eta} n_y, \quad \frac{\partial r}{\partial t} = \nabla r \cdot \mathbf{t} = -\frac{\partial r}{\partial \xi} n_y + \frac{\partial r}{\partial \eta} n_x$$

$$(4.14)$$

Expressions regarding partial derivatives of r along with their derivation may be found in Appendix A. The last of Eqs. (4.14) is obtained by noting that the components of the tangential unit vector \mathbf{t} are $t_x = -n_y$ and $t_y = n_x$. Attention should

be paid to the evaluation of the derivatives $\partial (\partial v / \partial n) / \partial x$ and $\partial (\partial v / \partial n) / \partial y$. The differentiation with respect to n is carried out at point $q(\xi, \eta) \in \Gamma$, while differentiation with respect to x or y is carried out at point $P(x, y) \in \Omega$.

Equations (4.12) and (4.13) are discretized in the same way as Eq. (4.1) and they yield the following expressions for the evaluation of the derivatives $u_{,x}$ and $u_{,y}$ at point P(x,y)

$$u_{x}(P) = \left(\frac{\partial u}{\partial x}\right)_{P} = \sum_{j=1}^{N} \left(\hat{H}_{Pj}\right)_{x} u^{j} - \sum_{j=1}^{N} \left(G_{Pj}\right)_{x} u_{n}^{j}$$
(4.15)

$$u_{y}(P) = \left(\frac{\partial u}{\partial y}\right)_{P} = \sum_{j=1}^{N} \left(\hat{H}_{Pj}\right)_{y} u^{j} - \sum_{j=1}^{N} \left(G_{Pj}\right)_{y} u^{j}_{n}$$
(4.16)

where the influence coefficients are given by the integrals

$$(G_{P_{j}})_{,x} = \int_{\Gamma_{j}} \frac{\partial v(P,q)}{\partial x} ds , \quad (\hat{H}_{P_{j}})_{,x} = \int_{\Gamma_{j}} \frac{\partial}{\partial x} \left[\frac{\partial v(P,q)}{\partial n_{q}} \right] ds$$

$$(G_{P_{j}})_{,y} = \int_{\Gamma_{j}} \frac{\partial v(P,q)}{\partial y} ds , \quad (\hat{H}_{P_{j}})_{,y} = \int_{\Gamma_{j}} \frac{\partial}{\partial y} \left[\frac{\partial v(P,q)}{\partial n_{q}} \right] ds$$

$$(4.17)$$

4.3 Evaluation of line integrals

The line integrals G_{ij} and \hat{H}_{ij} defined in Eq. (4.3) are evaluated numerically using a standard Gaussian quadrature. Of course, these integrals can be evaluated through symbolic languages, e.g. *Mathematica* [1] or *Maple* [2], but the resulting expressions are very lengthy and usually cover more than a page. Hence, the advantage of accuracy over the numerical integration is rather lost, due to the complexity of the mathematical expressions. For this reason, the Gaussian integration remains as the most suitable method for computing line integrals (see Appendix B). Two cases are distinguished for the integrals of the influence coefficients.

(i) Off-diagonal elements, $i \neq j$

In this case the point $p_i(x_i, y_i)$ lies outside the *j*-element, which means that the distance $r = |q - p_i|$ does not vanish and, consequently, the integral is regular.

The Gaussian integration is performed over the interval $-1 \le \xi \le 1$,

$$\int_{-1}^{1} f(\xi) d\xi = \sum_{k=1}^{n} w_k f(\xi_k)$$
(4.18)

where n is the number of integration points (Gauss points), and ξ_k and w_k (k = 1, 2, ..., n) are the abscissas and weights of the Gaussian quadrature of order n.

Let us consider the element j over which the integration will be carried out. This element is defined by the coordinates (x_j, y_j) and (x_{j+1}, y_{j+1}) of its extreme points, which are expressed in a *global system* having axes x and y, and origin at point O (Fig. 4.3). Subsequently, a *local system of axes* x' and y' is introduced at point p_j of the element. The local coordinates (x', 0) of point q on the j-th element are related to the global coordinates of the xy-system through the expressions

$$x = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{\ell_j} x'$$
(4.19a)

$$y = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{\ell_j} x', \qquad -\frac{\ell_j}{2} \le x' \le \frac{\ell_j}{2}$$
(4.19b)

where ℓ_j is the length of the *j*-th element and is given in terms of the coordinates of the end points as

$$\ell_{j} = \sqrt{(x_{j+1} - x_{j})^{2} + (y_{j+1} - y_{j})^{2}}$$



Figure 4.3 Global and local coordinate systems.

Expressions that map the global coordinates onto the integration interval [-1, +1] are obtained by introducing to Eqs. (4.19) the geometric relation

$$\frac{x'}{\ell_j/2} = \xi$$

Thus, the resulting coordinate transformation becomes

$$x(\xi) = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{2} \xi$$
(4.20a)

$$y(\xi) = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{2} \xi$$
(4.20b)

Moreover, it is

$$ds = \sqrt{dx^{2} + dy^{2}} = \sqrt{\left(\frac{x_{j+1} - x_{j}}{2}\right)^{2} + \left(\frac{y_{j+1} - y_{j}}{2}\right)^{2}} d\xi$$
$$= \frac{\ell_{j}}{2} d\xi$$
(4.21)

Hence, the Jacobian of the transformation is

$$|J(\xi)| = \frac{\ell_j}{2}$$

On the basis of the foregoing, the integrals of the influence coefficients are evaluated numerically in the following way:

(a) The integral of G_{ij}

$$G_{ij} = \int_{\Gamma_j} v \, ds = \int_{-1}^1 \frac{1}{2\pi} \, \ell n [r(\xi)] \frac{\ell_j}{2} \, d\xi = \frac{\ell_j}{4\pi} \sum_{k=1}^n \, \ell n [r(\xi_k)] \, w_k \tag{4.22}$$

where

$$r(\xi_k) = \sqrt{[x(\xi_k) - x_i]^2 + [y(\xi_k) - y_i]^2}$$
(4.23)

(b) The integral of \hat{H}_{ij}

This integral can also be evaluated analytically according to Refs. [3, 4]. Referring to Fig. 4.4, we notice that

$$ds \cos \phi = r \, d\alpha$$

which can be used along with Eqs. (4.14) and (A.7) to derive the expression

$$\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial v}{\partial n} ds = \int_{\Gamma_j} \frac{1}{2\pi} \frac{\cos \phi}{r} ds = \int_{\Gamma_j} \frac{1}{2\pi} da = \frac{a_{j+1} - a_j}{2\pi}$$
(4.24)

The angles a_{j+1} and a_j are computed from

$$\tan a_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} \tag{4.25}$$

$$\tan a_j = \frac{y_j - y_i}{x_j - x_i}$$
(4.26)

where x_{j+1} , y_{j+1} and x_j , y_j are the coordinates of the extreme points of the j-th element.



Figure 4.4 Definition of angles involved in the numerical integration over constant elements.

(ii) Diagonal elements, i = j

In this case the node p_i coincides with node p_j , and r lies on the element. Consequently, it is $\phi = \pi/2$ or $\phi = 3\pi/2$, which yields $\cos \phi = 0$. Moreover, we have

$$x_{p_j} = rac{x_{j+1} + x_j}{2}, \quad y_{p_j} = rac{y_{j+1} + y_j}{2}$$

and

$$r(\xi) = \sqrt{\left[x(\xi) - x_{p_j}\right]^2 + \left[y(\xi) - y_{p_j}\right]^2} = \frac{\ell_j}{2} |\xi|$$
(4.27)

Hence,

$$G_{jj} = \int_{\Gamma_j} \frac{1}{2\pi} \ell nr \, ds = 2 \int_0^{\ell_j/2} \frac{1}{2\pi} \ell nr \, dr$$

= $\frac{1}{\pi} \Big[r \, \ell nr - r \Big]_0^{\ell_j/2} = \frac{1}{\pi} \frac{\ell_j}{2} [\ell n \, (\ell_j/2) - 1]$ (4.28a)

and

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$$\hat{H}_{jj} = \frac{1}{2\pi} \int_{\Gamma_j} \frac{\cos \phi}{r} ds = \frac{1}{2\pi} \int_{-1}^{1} \frac{\cos \phi}{|\xi|} d\xi$$
$$= \frac{2}{2\pi} \Big[\cos \phi \, \ln |\xi| \Big]_{0}^{1} = 0$$
(4.28b)

It should be noted that for higher order elements (e.g. linear or parabolic) analytical integration is not applicable, and for this reason, other integration techniques are employed (see Chapter 5).

4.4 Evaluation of domain integrals

The integral representation of the solution for the Poisson equation (3.36) may be written for the boundary which has been discretized into N constant elements, as

$$\varepsilon^{i} u^{i} = \int_{\Omega} v f \, d\Omega - \sum_{j=1}^{N} \int_{\Gamma_{j}} v \frac{\partial u}{\partial n} ds + \sum_{j=1}^{N} \int_{\Gamma_{j}} u \frac{\partial v}{\partial n} ds \tag{4.29}$$

In some cases the techniques presented in Section 3.4 for transforming the domain integral

$$F' = \int_{\Omega} v(p_i, Q) f(Q) \, d\Omega_Q \tag{4.30}$$

to a boundary line integral are not suitable, and, if more advanced techniques are not utilized, the only recourse is a domain discretization. The domain Ω is divided into M two-dimensional elements or *cells*, e.g. triangular or rectangular cells (see Fig. 4.5), over which numerical integration is performed. Thus, employing twodimensional Gaussian integration, Eq. (4.30) becomes

$$F^{i} = \sum_{j=1}^{M} \left[\sum_{k=1}^{n} w_{k} v(p_{i}, Q_{k}) f(Q_{k}) \right] A_{j}$$
(4.31)

where Q_k and w_k (k = 1, 2, ..., n) are the k-th integration point and corresponding weight for the Gaussian quadrature on the j-th cell, and A_j (j = 1, 2, ..., M)is the area of the cell (see Appendix B).

By means of Eq. (4.31) and the notation introduced in Eqs. (4.3) and (4.5), Eq. (4.29) takes the following matrix form

$$[H]\{u\} + \{F\} = [G]\{u_n\}$$
(4.32)

Equations (4.32) are first reordered on the basis of the specified boundary conditions, and subsequently, they are solved for the unknown boundary quantities. The values of u at points inside Ω may then be computed from Eq. (4.29) for $\varepsilon^i = 1$. It should be noted that, for a point i (i = 1, 2, ..., M) lying on the j-th cell, the domain integral exhibits a logarithmic singularity and special care must be taken for its evaluation (see Appendix B and also Ref. [5]).


Figure 4.5 Discretization of domain Ω into triangular cells.

4.5 The Dual Reciprocity Method for Poisson's equation

The method presented in Section 3.5 for converting the domain integral

$$I(P) = \int_{\Omega} v(P,Q) f(Q) d\Omega_Q, \quad P(x,y) \in (\Omega \cup \Gamma), \ Q(x,y) \in \Omega$$
(4.33)

to a boundary integral, though effective, has an inherent drawback, which is the determination of function F as the solution of Eq. (3.52) for a given source density function f(x, y). Apparently, this procedure cannot be embedded in a computer code, since the user is responsible for providing this function and its normal derivative according to Eq. (3.55). This drawback can be overcome by employing the *Dual Reciprocity Method* (DRM) which was first introduced by Nardini and Brebbia [6] in 1982. They used the method to establish a consistent mass matrix in an effort to solve dynamic problems by utilizing the BEM and the static fundamental solution. Since then, the DRM was further developed to solve elliptic problems for which the fundamental solution could not be determined or was difficult to treat numerically, as well as to solve parabolic and hyperbolic problems employing simple static fundamental solutions. Nonlinear problems have also been attacked by this method. For a detailed presentation of the DRM, the interested reader is referred to the book by Partridge et al. [7]. A brief description of the DRM is given next.

Let us consider a set of M collocation points arbitrarily located in the domain Ω (see Fig. 4.6). According to the DRM the function f representing the source density is approximated by radial basis functions series, namely

$$f(Q(x,y)) = \sum_{j=1}^{M} a_j \phi_j(r_{jQ})$$
(4.34)

where a_j are M unknown coefficients and $\phi_j = \phi_j(r_{jQ})$ is a set of *radial basis functions* with argument

$$r_{jQ} = |Q - P_j| = \sqrt{(x - x_j)^2 + (y - y_j)^2}$$
(4.35)

which is the distance between the field point Q(x, y) and the collocation point $P_j(x_j, y_j)$. Although, the radial basis functions are usually denoted in BEM literature by $f_j(r)$, in the present book the notation $\phi_j(r)$ is preferred to avoid confusion with the source density function f. It should be noted that if the function f is defined on the boundary Γ , the total number of points employed in the series (4.34) can be chosen to be M + N instead of M, where N is the number of boundary nodal points used for the discretization of the boundary.



Figure 4.6 Integration over a plane domain Ω bounded by a curve Γ .

The unknown coefficients a_j are determined in the following way. First, we apply Eq. (4.34) to all collocation points obtaining expressions of the form

$$f_i = f(P_i) = \sum_{j=1}^{M} a_j \phi_j(r_{j_i}) \qquad (i = 1, 2, \dots, M)$$
(4.36)

where

$$r_{ji} = |Q_i - Q_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$
(4.37)

Using matrix notation Eq. (4.36) is written as

$$\{f\} = [\Phi]\{a\}$$
(4.38)

in which $\{f\}$ is a vector containing the values of function f at the M collocation points, $\{a\}$ is the vector of the M unknown coefficients and $[\Phi] = [\phi_j(r_{ji})]$ an $M \times M$ known matrix.

Assuming that $[\Phi]$ is not singular, Eq. (4.38) can be solved to yield

$$\{a\} = [\Phi]^{-1}\{f\}$$
(4.39)

Substituting Eq. (4.34) into Eq. (4.33) gives

$$I(P) = \sum_{j=1}^{M} a_j \left[\int_{\Omega} v(P,Q) \phi_j(r_{jQ}) d\Omega_Q \right]$$
(4.40)

The domain integrals in the above equation can be converted to boundary line integrals using the procedure presented in Section 3.5 (ii). For this purpose, we introduce first the functions $\hat{u}_j = \hat{u}_j(r)$ (j = 1, 2, ..., M), which are determined as the particular solution of the equations

$$\nabla^2 \hat{u}_j = \phi_j(r) \tag{4.41}$$

Note that the derivatives involved in the Laplace operator are taken with respect to point Q(x, y).

Subsequently, we apply Eq. (3.55) for the domain $\Omega^* \subseteq \Omega$ setting $f = \phi_j$ and $F = \hat{u}_j$, which yields the following expression

$$\int_{\Omega} v(P,Q) \phi_j(Q) d\Omega_Q = \int_{\Omega} v(P,Q) \nabla^2 \hat{u}_j(Q) d\Omega_Q$$
$$= \varepsilon(P) \hat{u}_j(P)$$
$$+ \int_{\Gamma} \left[v(P,q) \frac{\partial \hat{u}_j(q)}{\partial n_q} - \hat{u}_j(q) \frac{\partial v(P,q)}{\partial n_q} ds_q \right]$$
(4.42)

where $\varepsilon(P) = 1, \frac{1}{2}, 0$ depending on whether the point P lies inside Ω^* , on Γ^* or outside Ω^* , respectively.

Substituting the domain integral of Eq. (4.42) into Eq. (4.40), the latter becomes

$$I(P) = \sum_{j=1}^{M} a_j \left\{ \varepsilon(P) \,\hat{u}_j(P) + \int_{\Gamma} \left[v(P,q) \frac{\partial \hat{u}_j(q)}{\partial n_q} - \hat{u}_j(q) \frac{\partial v(P,q)}{\partial n_q} ds_q \right] \right\} \quad (4.43)$$

The boundary integrals in Eq. (4.43) are approximated by utilizing the same discretization as that adopted for the evaluation of the unknown boundary quantities (for the case $\Gamma^* = \Gamma$). Applying Eq. (4.43) for all the N boundary nodal points by letting point P to coincide with each of the nodal points, we obtain the vector $\{F\}$ of Eq. (4.32) as

$$\{F\} = \left([G][\hat{Q}] - [H][\hat{U}]\right)\{a\}$$
(4.44)

where [G] and [H] are the matrices defined in Section 4.2 and $[\hat{U}]$, $[\hat{Q}]$ are $N \times M$ matrices with entries

$$\hat{U}_{ij} = \hat{u}_j(r_{ji}) \tag{4.45a}$$

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$$\hat{Q}_{ij} = \frac{\partial \hat{u}_j(r_{ji})}{\partial n}$$
(4.45b)

where $i=1,2,\ldots,N$, $j=1,2,\ldots,M$ and

$$r_{ji} = |p_i - P_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad p_i \in \Gamma$$

Finally, combining Eqs. (4.39) and Eq. (4.44) we arrive at

$$\{F\} = [R]\{f\} \tag{4.46}$$

where

$$[R] = \left([G][\hat{Q}] - [H][\hat{U}] \right) [\Phi]^{-1}$$
(4.47)

Apparently, Eq. (4.46) states that the vector $\{F\}$ is expressed in terms of the values of the source density function f at the collocation points. This procedure is suitable when the function f is defined by a set of scattered values. It should be emphasized that the involvement of domain nodal points does not spoil the pure boundary character of BEM, since the discretization into elements and the integration are limited only to the boundary.

The success of DRM depends on the choice of the radial basis functions. Many types of radial basis functions have been reported in the literature, e.g. polynomial type ($\phi_j = 1 + r + r^2 + ...$), multiquadrics ($\phi_j = (r^2 + c^2)^{1/2}$ where c is an arbitrary constant), thin plate splines ($\phi_j = r^2 \ell nr$), etc. In general, the series of Eq. (4.34) must, first, converge to f in some sense for $M \to \infty$ and, second, ensure a good approximation of f for a relatively small number of collocation points. For more information about the radial basis functions the reader is advised to look in [8, 9, 10].

4.6 Program LABECON for solving the Laplace equation with constant boundary elements

On the basis of the analysis presented in the previous sections, a computer program has been written in FORTRAN language [11] for the solution of boundary value problems described by the Laplace equation. This program employs constant elements to approximate the line integrals.

Main program

The structure of program LABECON is described by the macro flow chart shown of Fig. 4.7. The main program defines the maximum dimensions of the arrays and opens two files. The file containing the data and the file in which the results are written. Next, it calls the following eight subroutines:

INPUT Reads the data from INPUTFILE.



Figure 4.7 Macro flow chart of program LABECON.

GMATR	Forms the matrix $[G]$ defined by Eq. (4.3).
HMATR	Forms the matrix $[H]$ defined by Eqs. (4.5) and (4.3).
ABMATR	Rearranges the matrices $[H]$ and $[G]$ according to the boundary conditions and forms the matrices $[A]$ and $\{B\}$ of Eq. (4.9).
SOLVEQ	Solves the system of linear equations $[A]{X} = {B}$ using Gauss elimination.
REORDER	Rearranges the boundary values and forms the vectors $\{u\}$ and $\{u_n\}$.
UINTER	Computes the values of u at the internal points.
OUTPUT	Writes the results in OUTPUTFILE.

The variables and arrays introduced in the program along with their meaning are given below:

N	Number of boundary elements and boundary nodes.							
IN	Number of internal points where the solution is computed.							
INDEX	One-dimensional array in which a type of boundary conditions is assigned to the nodes. It is $INDEX(J) = 0$ when u is prescribed, while $INDEX(J) = 1$ when $\partial u / \partial n$ is prescribed.							
XL	One-dimensional array containing the x coordinates of the extreme points of all the elements.							
YL	One-dimensional array containing the y coordinates of the extreme points of all the elements.							
ХМ	One-dimensional array containing the x coordinates of all the boundary nodes.							
YM	One-dimensional array containing the y coordinates of all the boundary nodes.							
G	Square matrix defined by Eq. (4.3).							
Н	Square matrix defined by Eqs. (4.5).							
A	Square matrix defined by Eq. (4.10a).							
UB	One-dimensional array. At input, it contains the boundary values of u , if INDEX(J) = 0, or $\partial u/\partial n$, if INDEX(J) = 1. At output it contains all the boundary nodal values of u .							
UNB	One-dimensional array containing the right hand side vector of equation $[A]{X} = \{B\}$ given by Eq. (4.10c). At the end it contains the boundary nodal values of $\partial u/\partial n$.							
XIN	One-dimensional array containing the x coordinates of the inter- nal points at which the values of u are computed.							

YIN	One-dimensional array containing the y coordinates of the inter- nal points at which the values of u are computed.
UIN	One-dimensional array containing the computed values of u at the internal points.

Subroutine INPUT

The subroutine INPUT reads in free FORMAT all the required data. The data have been written in INPUTFILE, to which the user has assigned a specific name as required by the main program. This file contains the following data:

- 1. User's name. One line with the name of the user.
- 2. *Title*. One line with the title of the program.
- 3. Extreme points of the boundary elements. N pairs of values consisting of the coordinates XL and YL of the extreme points. They are given in the positive sense, which for a closed domain Ω is counter-clockwise (Fig. 4.8a), whereas for an open domain (external domain) is clockwise (Fig. 4.8b).
- 4. Boundary conditions. N pairs of values consisting of either INDEX = 0 and u, or INDEX = 1 and $\partial u / \partial n$.
- 5. Internal points. IN pairs of values consisting of the coordinates XIN and YIN of the internal points at which the values of u will be computed.

Finally, the subroutine INPUT writes the data in OUTPUTFILE, to which the user assigns a specific name.



Figure 4.8 Positive direction and normal vector on the boundary of closed and open domains Ω .

Subroutine GMATR

The subroutine GMATR forms the matrix G defined in Eq. (4.3) by using the subroutines RLINTC and SLINTC. These subroutines perform the following tasks:

The RLINTC, (R)egular (L)ine (Int)egral for (C)onstant Elements, computes the off-diagonal elements of the G matrix.

The SLINTC, (S)ingular (L)ine (Int)egral for (C)onstant Elements, computes the diagonal elements of the G matrix.

Subroutine RLINTC

This subroutine computes regular line integrals on constant elements employing a four-point Gaussian integration scheme (see Appendix B). It uses the coordinates of the *i*-th nodal point and those of the extreme points of the *j*-th element (points *j* and j+1), it renames them to 0, 1 and 2 (see Fig. 4.9), respectively, and then it evaluates the integral of Eq. (4.22).



Figure 4.9 Four-point Gaussian quadrature and point numbering for subroutine RLINTC.

Subroutine SLINTC

This subroutine uses the coordinates of the extreme points of the j-th element (points j and j+1), it renames them to 1 and 2, respectively, and then calculates the integral on the basis of Eq. (4.28a).

Subroutine HMATR

It forms the matrix H defined in Eq. (4.5). The diagonal elements of the matrix \hat{H} are $\hat{H}_{ii} = 0$ (see Eq. 4.28b), while the off-diagonal elements are calculated using the subroutine DALPHA.

Subroutine DALPHA

This subroutine uses the coordinates of the i-th nodal point and those of the extreme points of the j-th element (points j and j+1), it renames them to 0, 1

and 2 (see Fig. 4.9), respectively, and then calculates the off-diagonal elements H_{ij} according to Eq. (4.24).

Subroutine ABMATR

This subroutine generates the matrix A and the vector B of Eq. (4.9) from the columns of matrices G and H. The J-th column of matrix A consists of the corresponding column of H, if u is unknown (INDEX(J) = 1), or that of -G, if u_n is unknown (INDEX(J) = 0). The vector B results as the sum of the columns of -Hmultiplied by the corresponding known values of u (INDEX(J) = 0) and the columns of G multiplied by the corresponding known values of u_n (INDEX(J) = 1). It should be noted that in the construction of A and B a change of sign occurs, when columns of H or G are transferred, respectively, to the left- or to the righthand side of the equation (see Eqs. 4.10).

Subroutine SOLVEQ

This subroutine employs the matrix A and the vector B (= UNB) and calls the subroutine LEQS, which solves the system of linear equations. The solution is stored in the vector UNB.

Subroutine LEQS

This subroutine uses the matrix A, the vector B and the parameter N to solve the system of equations AX = B by Gauss elimination. The solution is stored in the vector B. The output parameter LSING takes the value LSING = 0, when the matrix A is regular, or LSING = 1, when the matrix A is singular.

Subroutine REORDER

This subroutine rearranges the vectors UB and UNB on the basis of the boundary condition vector INDEX. At input the two vectors contain the known and the computed boundary values of u and u_n , respectively, while at output UB contains all the values of u and UNB all the values of its normal derivative $\partial u/\partial n$.

Subroutine UINTER

This subroutine uses the boundary values of the vectors UB and UNB to compute the values of u at the specified internal points on the basis of Eq. (4.11).

The evaluation of derivatives u_{x} and u_{y} at the internal points is left as an exercise to the reader (see Problem 4.1).

Subroutine OUTPUT

It prints all the results in the output file. First, it lists the coordinates of the boundary nodal points along with the corresponding nodal values of u and $\partial u/\partial n$, and then the coordinates of the internal points and the computed values of u at those points.

In the sequel, we provide the listing of program LABECON:

```
C
      PROGRAM LABECON
С
С
   This program solves the two dimensional (LA)place equation
С
   using the (B)oundary (E)lement method with (CON)stant
   boundary elements
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*15 INPUTFILE, OUTPUTFILE
С
С
   Set the maximum dimensions
C
      PARAMETER (N=16)
      PARAMETER (IN=9)
С
С
    N= Number of boundary elements which is equal to the number
       of boundary nodes
C
   IN= Number of internal points where the function u is calculated
C
C
      DIMENSION INDEX(N)
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), G(N,N), H(N,N)
      DIMENSION UB(N), A(N, N), UNB(N), XIN(IN), YIN(IN), UIN(IN)
С
С
   Read the names and open the input and output files
С
      WRITE (*, '(A)')' Name of the INPUTFILE (max.15 characters)'
      READ (*, '(A)') INPUTFILE
      WRITE (*,'(A)')' Name of the OUTPUTFILE (max.15 characters)'
           (*,'(A)') OUTPUTFILE
      READ
      OPEN (UNIT=1, FILE=INPUTFILE)
      OPEN (UNIT=2, FILE=OUTPUTFILE)
С
С
   Read the data from INPUTFILE
С
      CALL INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN)
С
C
   Compute the G matrix
C
      CALL GMATR (XL, YL, XM, YM, G, N)
С
С
   Compute the H matrix
С
      CALL HMATR (XL, YL, XM, YM, H, N)
С
С
   Form the system of equations AX=B
С
      CALL ABMATR (G, H, A, UNB, UB, INDEX, N)
C
   Solve the system of equations
С
C
      CALL SOLVEQ(A, UNB, N, LSING)
C
   Form the vectors U and UN of all the boundary values
С
С
      CALL REORDER (UB, UNB, INDEX, N)
С
С
   Compute the values UIN of u at the internal points
С
      CALL UINTER (XL, YL, XIN, YIN, UB, UNB, UIN, N, IN)
С
   Print the results in the OUTPUTFILE
С
C
      CALL OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, N, IN)
C
```

```
C Close input and output files
С
      CLOSE(1)
      CLOSE(2)
С
      STOP
      END
С
С
C_____
С
      SUBROUTINE INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN)
С
  This subroutine reads the data from the input file
С
С
  and writes them in the output file
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*80 NAME, TITLE
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), INDEX(N), UB(N)
С
      WRITE(2,100)
  100 FORMAT('',69('*'))
С
С
  Read user's name
С
     READ(1, '(A)')NAME
С
      WRITE(2, '(A) ')NAME
С
  Read the title of the program
C
С
     READ(1, '(A)')TITLE
С
      WRITE(2, '(A) ')TITLE
С
      WRITE(2,200)N, IN
  200 FORMAT(//'BASIC PARAMETERS'//2X, 'NUMBER OF BOUNDARY ELEMENTS='
     1,13/2X, 'NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED
     1 = ', I3
С
  Read the coordinates XL, YL of the extreme points of the boundary elements
С
С
      READ(1,*) (XL(I),YL(I),I=1,N)
С
  Write the coordinates in the output file
C
C
      WRITE(2,300)
  300 FORMAT(//2X, 'COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY '
     l'elements',//2X, 'POINT',9X, 'XL',15X, 'YL')
      DO 20 I=1,N
   20 WRITE(2,400) I,XL(I),YL(I)
  400 FORMAT(2X,13,2(3X,E14.5))
С
  Read the boundary conditions and store them in UB(I).
С
  If INDEX(I)=0, the value UB(I) is the prescribed value of u.
С
С
  If INDEX(I) =1, the normal derivative un is prescribed.
С
      READ(1,*) (INDEX(I),UB(I),I=1,N)
С
  Write the boundary conditions in the output file
С
C
      WRITE(2,500)
  500 FORMAT(//2X, 'BOUNDARY CONDITIONS'//2X, 'NODE', 6X, 'INDEX',
     1 7X, 'PRESCRIBED VALUE')
      DO 30 I=1,N
```

```
30 WRITE(2,600) I, INDEX(I), UB(I)
  600 FORMAT (2X, I3, 9X, I1, 8X, E14.5)
С
С
  Read the coordinates of the internal points
С
     READ(1,*) (XIN(I),YIN(I),I=1,IN)
     RETURN
     END
С
С
C=
       С
     SUBROUTINE GMATR (XL, YL, XM, YM, G, N)
С
С
  This subroutine computes the elements of the G matrix
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XL (N+1), YL (N+1), XM (N), YM (N)
     DIMENSION G(N,N)
C
  Compute the nodal coordinates and store them in the arrays
С
С
  XM and YM
С
     XL(N+1) = XL(1)
     YL(N+1) = YL(1)
     DO 10 I=1,N
     XM(I) = (XL(I) + XL(I+1))/2.
  10 YM(I) = (YL(I) + YL(I+1))/2.
С
С
  Compute the elements of matrix G
С
     DO 20 I=1,N
     DO 20 J=1,N
     JP1=J+1
     IF(I.NE.J)THEN
     CALL RLINTC(XM(I),YM(I),XL(J),YL(J),XL(JP1),YL(JP1),RESULT)
     G(I,J) = RESULT
     ELSEIF (I.EQ.J) THEN
     CALL SLINTC(XL(J),YL(J),XL(JP1),YL(JP1),RESULT)
     G(I,J)=RESULT
     ENDIF
   20 CONTINUE
     RETURN
     END
С
С
С
      SUBROUTINE RLINTC(X0,Y0,X1,Y1,X2,Y2,RESULT)
с
  This subroutine computes the off-diagonal elements of the
С
C
  matrix G
С
C
  RA= The distance of point 0 from the Gauss integration point
с
      on the boundary element
С
С
  WG= The weights of the Gauss integration
С
С
  XI= The coordinates of the Gauss integration points in the
C
      interval [-1,1]
С
С
  XC,YC= The global coordinates of the Gauss integration points
с
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XC(4), YC(4), XI(4), WG(4)
```

```
DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      PI=ACOS(-1.)
      AX = (X2 - X1) / 2.
      AY = (Y2 - Y1) / 2.
      BX = (X2 + X1) / 2.
      BY = (Y2 + Y1) / 2.
С
С
  Compute the line integral
С
      RESULT=0.
С
      DO 30 I=1,4
      XC(I) = AX * XI(I) + BX
      YC(I) = AY * XI(I) + BY
      RA=SQRT((XC(I)-X0)**2+(YC(I)-Y0)**2)
   30 RESULT=RESULT+DLOG(RA) *WG(I)
      SL=2.*SQRT(AX**2+AY**2)
      RESULT=RESULT*SL/(4.*PI)
      RETURN
      END
с
С
C==
    С
      SUBROUTINE SLINTC(X1,Y1,X2,Y2,RESULT)
С
С
  This subroutines computes the diagonal elements of the matrix G
С
     IMPLICIT REAL*8 (A-H,O-Z)
     PI=ACOS(-1.)
     AX = (X2 - X1) / 2.
     AY = (Y2 - Y1) / 2.
     SL=SQRT(AX**2+AY**2)
     RESULT=SL*(DLOG(SL)-1.)/PI
     RETURN
     END
С
С
C==
          С
     SUBROUTINE HMATR(XL,YL,XM,YM,H,N)
С
С
  This subroutine computes the elements of the H matrix
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XL(N+1), YL(N+1), XM(N), YM(N)
     DIMENSION H(N,N)
С
     PI=ACOS(-1.)
С
     XL(N+1) = XL(1)
     YL(N+1) = YL(1)
     DO 10 I=1,N
      XM(I) = (XL(I) + XL(I+1))/2.
  10 YM(I) = (YL(I) + YL(I+1))/2.
С
С
  Compute the elements of the H matrix
С
     DO 20 I=1,N
     DO 20 J=1,N
      IF (I.NE.J) THEN
      CALL DALPHA (XM(I),YM(I),XL(J),YL(J),XL(J+1),YL(J+1),RESULT)
     H(I,J) = RESULT
      ELSEIF (I.EQ.J) THEN
```

```
H(I, J) = -0.5
      ENDIF
   20 CONTINUE
      RETURN
      END
С
С
C=====
       ****
С
      SUBROUTINE DALPHA (X0, Y0, X1, Y1, X2, Y2, RESULT)
C
С
   This subroutine computes the off diagonal elements of
С
   the H matrix
С
      IMPLICIT REAL*8 (A-H,O-Z)
      PI=ACOS(-1.)
      DY1=Y1-Y0
      DX1=X1-X0
     DY2=Y2-Y0
     DX2 = X2 - X0
     DL1=SQRT(DX1**2+DY1**2)
      COS1=DX1/DL1
      SIN1=DY1/DL1
     DX2R=DX2*COS1+DY2*SIN1
     DY2R=-DX2*SIN1+DY2*COS1
     DA=ATAN2 (DY2R, DX2R)
     RESULT=DA/(2.*PI)
     RETURN
     END
C
С
C_________
С
     SUBROUTINE ABMATR (G, H, A, UNB, UB, INDEX, N)
С
с
  This subroutine rearranges the matrices G and H and produces
С
   the matrices A and B≈UNB
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION G(N,N), H(N,N), A(N,N), UNB(N), UB(N), INDEX(N)
С
  Reorder the columns of the system of equations and
С
с
  store them in A
С
     DO 40 J=1,N
     IF(INDEX(J).EQ.0) THEN
     DO 20 I=1,N
   20 A(I,J) = -G(I,J)
     ELSEIF(INDEX(J).NE.0) THEN
     DO 30 I=1,N
   30 A(I,J) = H(I,J)
     END IF
   40 CONTINUE
С
С
  Compute the right hand side vector and store it in UNB
с
     DO 50 I=1,N
     \text{UNB}(1) = 0.
     DO 60 J=1,N
     IF(INDEX(J).EQ.0) THEN
     UNB(I) = UNB(I) - H(I, J) * UB(J)
     ELSEIF(INDEX(J).NE.0)THEN
     UNB(I) = UNB(I) + G(I, J) * UB(J)
     ENDIF
   60 CONTINUE
```

```
50 CONTINUE
     RETURN
     END
С
С
C===
         с
      SUBROUTINE SOLVEQ(A, UNB, N, LSING)
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION A(N,N), UNB(N)
     CALL LEQS(A, UNB, N, LSING)
     IF (LSING.EQ.0) THEN
     WRITE(2,150)
  150 FORMAT(/,'',69('*')//2X'The system has been solved regularly'/)
     ELSEIF(LSING.EQ.1) THEN
     WRITE(2.170)
  170 FORMAT(/,'',69('*')//2X'The system is singular'/)
     ENDIF
     RETURN
     END
Ċ
С
C==
      С
     SUBROUTINE LEQS(A, B, N, LSING)
С
с
   This subroutine uses Gauss elimination to solve
С
   a system of linear equations, [A] \{X\} = \{B\}, where
   A : One-dimensional array which contains the occasional row of
С
       the two-dimensional array of the coefficients of the unknowns
С
С
   B : One-dimensional array which contains the known coefficients
   N : Integer denoting the number of the unknowns
С
С
   LSING: Integer taking the values:
С
         LSING = 0, if the system has been solved regularly
С
         LSING = 1, if the system is singular
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION A(1), B(1)
С
     LSING=0
     JJ = -N
     DO 10 J=1,N
       JY=J+1
       JJ=JJ+N+1
       AMAX = 0.0
       IHELP=JJ-J
     DO 20 I=J,N
       IJ=IHELP+I
       IF (ABS (AMAX) - ABS (A (IJ))) 30,20,20
  30
          AMAX=A(IJ)
          TMAX=T
  20 CONTINUE
        IF (ABS (AMAX) .EQ.0.) THEN
           LSING=1
           RETURN
        END IF
     I1=J+N*(J-2)
     IHELP=IMAX-J
     DO 40 K=J,N
       Il=Il+N
       I2=I1+IHELP
       ATEMP=A(I1)
       A(I1) = A(I2)
       A(I2)=ATEMP
```

```
40 A(I1) = A(I1) / AMAX
      ATEMP=B(IMAX)
      B(IMAX) = B(J)
      B(J) = ATEMP / AMAX
         IF(J-N) 50,70,50
   50
            IQS=N*(J-1)
            DO 10 IX=JY,N
              IXJ=IQS+IX
              IHELP=J-IX
              DO 60 JX=JY,N
                IJREF=N*(JX-1)+IX
                JJX = IJREF + IHELP
   60
                A(IJREF) = A(IJREF) - (A(IXJ) * A(JJX))
   10 B(IX) = B(IX) - B(J) * A(IXJ)
   70
        NY = N - 1
      NN=N*N
      DO 80 J=1,NY
        I1=NN-J
        I2=N-J
       T3=N
      DO 80 K=1,J
        B(I2) = B(I2) - A(I1) * B(I3)
        I1=I1-N
   80
       I3=I3-1
      RETURN
      END
с
C
C=
                          С
      SUBROUTINE REORDER (UB, UNB, INDEX, N)
Ç
С
  This subroutine rearranges the arrays UB and UNB in such a way
C that all values of the function u are stored in UB while all
C the values of the normal derivative un are stored in UNB
С
      IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION UB(N), UNB(N), INDEX(N)
С
     DO 20 I=1,N
     IF(INDEX(I))20,20,10
   10 CH=UB(I)
     UB(I) = UNB(I)
      UNB(I) = CH
   20 CONTINUE
     RETURN
      END
С
С
C = =
      С
      SUBROUTINE UINTER (XL, YL, XIN, YIN, UB, UNB, UIN, N, IN)
С
С
  This subroutine computes the values of u at the internal points
С
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), UB(N), UNB(N), UIN(IN)
C
С
  Compute the values of u at the internal points
С
     DO 10 K=1, IN
     UIN(K) = 0.
     DO 20 J=1,N
      JP1=J+1
      CALL DALPHA (XIN(K), YIN(K), XL(J), YL(J), XL(JP1), YL(JP1), RESH)
```

```
CALL RLINTC(XIN(K), YIN(K), XL(J), YL(J), XL(JP1), YL(JP1), RESG)
   20 UIN(K)=UIN(K)+RESH*UB(J)-RESG*UNB(J)
   10 CONTINUE
     RETURN
      END
С
С
С
      SUBROUTINE OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, N, IN)
С
С
  This subroutine prints the results in the output file.
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XM(N), YM(N), UB(N), UNB(N)
     DIMENSION XIN(IN), YIN(IN), UIN(IN)
С
     WRITE(2,100)
  100 FORMAT('',69('*')///1X,'RESULTS'///2X,'BOUNDARY NODES'//
    1 11X, 'X', 15X, 'Y', 15X, 'U', 15X, 'Un'/)
с
     DO 10 I=1,N
  10 WRITE(2,200) XM(I), YM(I), UB(I), UNB(I)
  200 FORMAT(4(2X,E14.5))
С
     WRITE (2,300)
  300 FORMAT(//,2X,'INTERNAL POINTS'//10X,'X',15X,'Y',11X,
    1 'SOLUTION U'/)
С
     DO 20 K=1, IN
  20 WRITE(2,400) XIN(K), YIN(K), UIN(K)
  400 FORMAT(3(2X,E14.5))
     WRITE(2,500)
  500 FORMAT(/,'',69('*'))
     RETURN
     END
с
С
C = =
```

Example 4.1

The scope of this example is to illustrate the use of program LABECON by solving a simple potential problem. The domain Ω is a square with mixed boundary conditions (Fig. 4.10). The boundary is discretized into 16 constant elements and the solution is sought at 9 internal points (Fig. 4.11).

The exact solution is: u(x,y) = 100(1+x).

In this example the number of elements (N = 16) is relatively small and thus the data file may be created manually. However, if a rectangular domain is discretized into a large number of elements, one should form the data file by using the program RECT-1.FOR included in the CD-ROM which is provided with the book.



Figure 4.10 Square domain Ω and boundary conditions of Example 4.1.



Figure 4.11 Boundary element discretization and internal points of Example 4.1.

EXAMPLE 4.1 (DATA)

J.T. KATSIKADELIS Example 4.1

	.00000000		.00000000
	.25000000		.00000000
	.50000000		.00000000
	.75000000		.00000000
	1.00000000		.00000000
	1.00000000		.25000000
	1.00000000		.50000000
	1.00000000		.75000000
	1.00000000		1.00000000
	.75000000		1.00000000
	.50000000		1.00000000
	.25000000		1.00000000
	.00000000		1.00000000
	.00000000		.75000000
	.00000000		.50000000
	.00000000		.25000000
_			
1	.0000	0000	
1	.0000	0000	
1	.0000	0000	
1	.0000	0000	
0	200.0000	0000	
0	200.0000	0000	
0	200.0000	00000	
1	200.0000	0000	
1	.0000	0000	
1	.0000	0000	
1	.0000	0000	
ā	100.0000	0000	
0	100.0000	0000	
ő	100 0000	0000	
0	100.0000	0000	
Ŭ	100.0000		
	0.25 0	. 25	

0.50	0.25
0.75	0.25
0.25	0.50
0.50	0.50
0.75	0.50
0.25	0.75
0.50	0.75
0.75	0.75

EXAMPLE 4.1 (RESULTS)

BASIC PARAMETERS

NUMBER OF BOUNDARY ELEMENTS= 16 NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED= 9

COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELEMENTS

 POINT
 XL
 YL

 1
 .00000E+00
 .00000E+00

2	.25000E+00	.00000E+00
3	.50000E+00	.00000E+00
4	.75000E+00	.00000E+00
5	.10000E+01	.00000E+00
6	.10000E+01	.25000E+00
7	.10000E+01	.50000E+00
8	.10000E+01	.75000E+00
9	.10000E+01	.10000E+01
10	.75000E+00	.10000E+01
11	.50000E+00	.10000E+01
12	.25000E+00	.10000E+01
13	.00000E+00	.10000E+01
14	.00000E+00	.75000E+00
15	.00000E+00	.50000E+00
16	.00000E+00	.25000E+00

BOUNDARY CONDITIONS

NODE	INDEX	PRESCRIBED VALUE
1	1	.00000E+00
2	1	.00000E+00
3	1	.00000E+00
4	1	.00000E+00
5	0	.20000E+03
6	0	.20000E+03
7	0	.20000E+03
8	0	.20000E+03
9	1	.00000E+00
10	1	.00000E+00
11	1	.00000E+00
12	1	.00000E+00
13	0	.10000E+03
14	0	.10000E+03
15	0	.10000E+03
16	0	.10000E+03

The system has been solved regularly

RESULTS

BOUNDARY NODES

x	Y	υ	Un
.12500E+00	.00000E+00	.11188E+03	.00000E+00
.37500E+00	.00000E+00	.13732E+03	.00000E+00
.62500E+00	.00000E+00	.16268E+03	.00000E+00
.87500E+00	.00000E+00	.18812E+03	.00000E+00
.10000E+01	.12500E+00	.20000E+03	.10552E+03
.10000E+01	.37500E+00	.20000E+03	.98417E+02
.10000E+01	.62500E+00	.20000E+03	.98417E+02
.10000E+01	.87500E+00	.20000E+03	.10552E+03
.87500E+00	.10000E+01	.18812E+03	.00000E+00
.62500E+00	.10000E+01	.16268E+03	.00000E+00
.37500E+00	.10000E+01	.13732E+03	.00000E+00
.12500E+00	.10000E+01	.11188E+03	.00000E+00
.00000E+00	.87500E+00	.10000E+03	10552E+03
.00000E+00	.62500E+00	.10000E+03	98417E+02

.00000E+00 .00000E+00	.37500E+00 .12500E+00	.10000E+03 .10000E+03	98417E+02 10552E+03
INTERNAL POINTS			
x	Y	SOLUTION U	
.25000E+00	.25000E+00	.12489E+03	
.50000E+00	.25000E+00	.15000E+03	
.75000E+00	.25000E+00	.17511E+03	
.25000E+00	.50000E+00	.12495E+03	
.50000E+00	.50000E+00	.15000E+03	
.75000E+00	.50000E+00	.17505E+03	
.25000E+00	.75000E+00	.12489E+03	
.50000E+00	.75000E+00	.15000E+03	
.75000E+00	.75000E+00	.17511E+03	

Table 4.1	Computed boundary and internal values for various boundary
	discretizations of Example 4.1.

Point		Fract						
Tom	16	48	80	112	144	Exact		
Values of u at the boundary nodes								
1	111.88	112.36	112.43	112.46	112.47	112.50		
2	137.32	137.47	137.48	137.49	137.49	137.50		
3	162.68	162.53	162.52	162.51	162.51	162.50		
4	188.12 187.64 187.57 187.54 187.53					187.50		
		Values of	u_n at the be	oundary not	les			
5	105.520	98.215	99.486	99.665	99.774	100.000		
6	98.417	99.800	99.909	99.946	99.964	100.000		
	Values of u at the internal points							
1	124.89	124.98	124.99	124.99	125.00	125.00		
5	150.00	150.00	150.00	150.00	150.00	150.00		
9	175.11	175.02	175.01	175.01	175.00	175.00		

As it was anticipated, the obtained solution is symmetric with respect to the axis passing through the center of the square and being parallel to the x-axis. Table 4.1 presents the computed values on the boundary and at the interior of the domain versus the number N of boundary elements. Comparing these results to the exact values, it is apparent that they converge rapidly and the computed boundary values of u and u_n are very close to the exact ones. The accuracy at the internal points is

even better, being attributed to the fact that these values are computed from Eq. (4.11), which is a weighted residual form for all the boundary values. The results have been obtained using the Microsoft Fortran PowerStation on a PC. The running time was a few seconds. Actually, the time required to solve the problem is the time needed to prepare the data file. For this reason, the user of LABECON is advised to write first a simple program that generates the coordinates XL, YL of the extreme points and XIN, YIN of the internal points, as well as the values of the INDEX vector and the boundary conditions. In this way, the tedious task of entering data by hand and the ensuing possible errors are avoided.

Example 4.2

The scope of this example is to demonstrate the efficiency of program LABECON in treating domains with curvilinear boundaries. Specifically, we want to compute u for the following Neumann problem:

$$abla^2 u = 0 \quad \text{in } \Omega$$
 $\frac{\partial u}{\partial n} = \overline{u}_n \quad \text{on } \Gamma$

where the domain Ω is the ellipse shown in Fig. 4.12 and

$$\overline{u}_n = \frac{2(b^2 x^2 - a^2 y^2)}{\sqrt{b^4 x^2 + a^4 y^2}}$$
(a)

The exact solution is known to be

$$u = x^2 - y^2 + C \tag{b}$$



Figure 4.12 Elliptic domain Ω of Example 4.2.

Practically, the discretization of the boundary into elements of equal length is not a simple problem. Besides, such a discretization would not give the best results, since the curvature of the boundary is more intense in the neighborhood of point A(5.0,0.0) and decreases moving towards point B(0.0,3.0). A better discretization would have elements whose length is small near A and increases progressively towards B. This discretization can be achieved by establishing the extreme points of the element from the parametric equations of the ellipse for constant increment $\Delta\theta$ of the parameter. Namely, the coordinates are given as

 $x_i = a \cos \theta_i$, $y_i = b \sin \theta_i$ $(i = 1, 2, \dots, N)$

where N is the total number of boundary elements, and



Figure 4.13 Boundary element discretization and internal points of Example 4.2.

For example, a discretization of the boundary into N = 20 elements (Fig. 4.13) produces the following lengths for elements 1 through 5 of the first quadrant

 $\ell_1 = 0.958$, $\ell_2 = 1.097$, $\ell_3 = 1.289$, $\ell_4 = 1.457$, $\ell_5 = 1.552$

and in each of the other three quadrants the elements have the same lengths due to the symmetry with respect to the x and y axes.

The coordinates of the twelve internal points are computed similarly from the expressions

$$x_i = \frac{a}{2}\cos\theta_i , \quad y_i = \frac{b}{2}\sin\theta_i \qquad (i = 1, 2, ..., 12)$$

$$\theta_i = (i-1)\Delta\theta, \quad \Delta\theta = \frac{2\pi}{12}$$

As it has already been mentioned, the preparation of the data file by keying in the data, besides the risk of error, is a painful and time consuming process. Since LABECON is not a commercial computer code, it does not offer a pre-processing interface for automatic preparation of the data. To facilitate this procedure and to reduce possible errors a FORTRAN program has been written that generates the required data and stores them in the INPUTFILE. The program has been named ELLIPSE-1 and its listing is shown below.

```
С
      PROGRAM ELLIPSE1
с
С
  This program creates the INPUTFILE for LABECON when the
C domain is an ellipse
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*15 INPUTFILE
      CHARACTER*80 NAME, TITLE
      PARAMETER (N=20)
      PARAMETER (IN=13)
      DIMENSION XL(N+1), YL(N+1), INDEX(N), XIN(IN), YIN(IN)
с
     WRITE (*,'(A)')' Name of the INPUTFILE (max.15 characters)'
      READ (*, '(A) ') INPUTFILE
      OPEN (1, FILE=INPUTFILE)
     WRITE (*, '(A)')' User NAME (max.15 characters)'
     READ(*, '(A) ')NAME
     WRITE(1, '(A) ')NAME
     WRITE (*, '(A)')' Program TITLE (max.15 characters)'
     READ(*, '(A) ') TITLE
     WRITE(1, '(A)')TITLE
С
      WRITE(1,100)
  100 FORMAT(' ',79(' '))
С
      A=5.
      B=3.
      PI=ACOS(-1.)
      DTHETA=(2.*PI)/N
     DO 1 I=1,N
      THETA=(I-1) *DTHETA
      XL(I) = A*COS(THETA)
      YL(I) = B*SIN(THETA)
      WRITE(1,200)XL(I),YL(I)
  200 FORMAT(2(2X,F14.7))
    1 CONTINUE
с
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
     WRITE(1,100)
С
      DO 2 I=1,N-1
      INDEX(I) = 1
      XM = (XL(I) + XL(I+1))/2.
      YM = (YL(I) + YL(I+1))/2.
      UNB=2.*((B*XM)**2-(A*YM)**2)/SQRT(B**2*(B*XM)**2+A**2*(A*YM)**2)
      WRITE(1,300) INDEX(I), UNB
```

```
2 CONTINUE
  300 FORMAT (2X, I4, 2X, F14.7)
     INDEX(N) = 0
     XM = (XL(N) + XL(N+1)) / 2.
     YM = (YL(N) + YL(N+1)) / 2.
     UB=XM**2-YM**2
     WRITE (1,300) INDEX(N), UB
     WRITE (1,100)
С
     DA=A/2.
     DB=B/2.
     WRITE(1,100)
С
     DTHETA=(2.*PI)/12.
     DO 3 I=1,12
     THETA= (I-1) *DTHETA
     XIN(I) = DA*COS(THETA)
     YIN(I)=DB*SIN(THETA)
     WRITE(1,200)XIN(I),YIN(I)
   3 CONTINUE
C
     XIN(IN) = 0.
     YIN(IN) = 0.
     WRITE (1,200) XIN(IN), YIN(IN)
С
     STOP
     END
С
С
```

EXAMPLE 4.2 (DATA)

J	•	т	•	ĸ	AT	S	I	ĸ	AE	E	L	I	s
E	x	a	mp	1	е	4		2					

5.0000000	.0000000
4.7552826	.9270510
4.0450850	1.7633558
2.9389263	2.4270510
1.5450850	2.8531695
.000000	3.0000000
-1.5450850	2.8531695
-2.9389263	2.4270510
-4.0450850	1.7633558
-4.7552826	.9270510
-5.0000000	.0000000
-4.7552826	9270510
-4.0450850	-1.7633558
-2.9389263	-2.4270510
-1.5450850	-2.8531695
.0000000	-3.0000000
1.5450850	-2.8531695
2.9389263	-2.4270510
4.0450850	-1.7633558
4.7552826	9270510
1 9.1955756	5
1 4.9664670	5
1 .0000000	C
1 -3.7385842	2
1 -5.680745	8

1	-5.68	07458
1	-3.73	85842
1	.00	00000
1	4.96	64676
1	9.19	55756
1	9.19	55756
1	4.96	64676
1	.00	00000
1	-3.73	85842
1	-5.68	07458
1	-5.68	07458
1	-3.73	85842
1	.000	00000
1	4.96	54676
0	23.57	65287
	2.5000000	.0000000
	2.1650635	.7500000
	1.2500000	1.2990381
	.0000000	1.5000000
-	1.2500000	1.2990381
-	2.1650635	.7500000
-	2.5000000	.0000000
-	2.1650635	7500000
-	1.2500000	-1.2990381
	.0000000	-1.5000000
	1.2500000	-1.2990381
	2.1650635	7500000
	.0000000	.0000000

Referring to the Neumann problem of Example 4.2, the following should be taken into account, which are known from the theory of partial differential equations of the elliptic type [12].

(a) In order for the problem to have a solution, it must be

$$\int_{\Gamma} \frac{\partial u}{\partial n} ds = 0$$

which can be readily proved by applying Eq. (2.16) for v = 1 and $\nabla^2 u = 0$.

(b) The solution u is not determined uniquely, but to the approximation of an arbitrary constant.

The first remark demands the following check

$$\int_{\Gamma} \overline{u}_n ds \approx \sum_{i=1}^N \ell_i (\overline{u}_n)_i \approx 0$$

which can be shown that is satisfied for the give data. The second remark produces difficulties in solving Eq. (4.9). The matrix [A] = [H] is singular in this case and therefore, it can not be inverted. This difficulty can be overcome by prescribing arbitrarily the value of u at a node, say at the last node N, and then solving the problem with mixed boundary conditions. Without restriction of generality, we

assign to u^N the exact value of u as it is computed from equation (b) with C = 0, so that the obtained numerical results can be directly compared with the exact ones.

In Table 4.2, the computed values of u at internal points are presented for various values of N. Moreover, Fig. 4.14 shows the variation of error in u at the internal point 2 versus the number of boundary elements N. In this problem, the convergence is slower than in Example 4.1. This was anticipated, because the curved boundary of the ellipse is approximated by an inscribed polygon. A faster convergence can be achieved by employing another type of constant element which approximates the geometry with a parabolic arc [13].

Table 4.2Computed values of u at the internal points for various boundary
discretizations of Example 4.2.

Point	Number of boundary elements, N					Exact	
	20	40	100	300	700	1000	Exact
1	7.1022	6.5060	6.2960	6.2556	6.2511	6.2506	6.2500
2	5.0959	4.4114	4.1759	4.1312	4.1262	4.1256	4.1250
3	1.0736	0.2201	-0.0646	-0.1177	-0.1236	-0.1243	-0.1250
4	-0.9207	-1.8715	-2.1842	-2.2422	-2.2485	-2.2492	-2.2500
13	1.2216	0.3514	0.0613	0.0073	0.0014	0.0007	0.0000



Figure 4.14 Variation of error in u at internal point 2 versus the number of boundary elements of Example 4.2.

4.7 Domains with multiple boundaries

In many applications the domain Ω may contain holes. In that case the contours are more than one (Fig. 4.15). Namely, there is an outer boundary enclosing a finite number of nonintersecting inner contours. In mathematical terms, this type of domain is referred to as *multiply connected domain*. We come across these domains in many problems such as torsion of bars with hollow cross-sections, fluid flow past obstacles or heat conduction in pipes with thermal insulation.



Figure 4.15 Multiply connected domain Ω .

It can be easily shown that Green's identity (2.16) is valid also for multiply connected domains, where the boundary Γ is the sum (union) of all the contours. Indeed, if we introduce the cuts AB and CD, which are arbitrary and not necessarily along straight lines (see Fig. 4.16), the domain Ω is converted to simply connected, that is one without holes. Green's identity (2.16) may then be written as

$$\begin{split} \int_{\Omega} \Big(v \, \nabla^2 u - u \, \nabla^2 v \Big) d\Omega &= \sum_{k=1}^3 \int_{\Gamma_k} \Big(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \Big) ds + \int_{BA} \Big(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \Big) ds \\ &+ \int_{AB} \Big(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \Big) ds + \int_{DC} \Big(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \Big) ds \\ &+ \int_{CD} \Big(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \Big) ds \end{split}$$

Noting that

$$\int_{BA} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = -\int_{AB} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$
$$\int_{DC} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = -\int_{CD} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$



Figure 4.16 Positive direction and normal vector on the boundaries of a multiply connected domain.

the foregoing equation becomes

$$\int_{\Omega} \left(v \, \nabla^2 u - u \, \nabla^2 v \right) d\Omega = \int_{\Gamma} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \tag{4.48}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Thus, Green's identity (2.16) applies also to multiply connected domains where the boundary integral is taken on all the contours. Consequently, relations resulting from Green's identity are valid for multiple boundaries as well. It should be noted that the positive sense on the inner contours is clockwise, which is opposite to that on the outer contours (Fig. 4.16).

4.8 Program LABECONMU for domains with multiple boundaries

The program LABECON can be readily modified to solve potential problems in domains with holes (multiply connected domains). The changes affect only the main program and the subroutines INPUT, GMATR, HMATR and UINTER. The new program has been given the name LABECONMU to distinguish it from LABECON.

The structure of LABECONMU is the same as that of LABECON shown in the macro flow chart of Fig. 4.7. In its main program two new parameters have been introduced, NB that defines the number of boundaries, and the vector NL(I), which identifies the number of the last element on the I-th boundary (I=1,2,...,NB). It should also be noted that the elements of all the boundaries are numbered consecutively and therefore, N denotes the total number of elements. The listings

of the main program as well as of the subroutines INPUT, GMATR, HMATR and UINTER are presented below:

```
С
      PROGRAM LABECONMU
C
   This program solves the two dimensional (LA)place equation
С
С
   using the (B)oundary (E)lement method with (CON)stant
С
   boundary elements for domains with (MU) ltiple boundaries
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*15 INPUTFILE, OUTPUTFILE
С
С
   Set the maximum dimensions
С
      PARAMETER (N=24)
      PARAMETER (IN=8)
      PARAMETER (NB=2)
С
   N= Number of boundary elements which is equal to the number
С
С
       of boundary nodes
С
  IN= Number of internal points where the function u is calculated
  NB= Number of boundaries of the multiple boundary domain
С
  NL= Array with entries the number of the last element on each
С
С
      boundary
С
     DIMENSION INDEX(N), NL(NB)
     DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), G(N,N), H(N,N)
     DIMENSION UB(N), A(N, N), UNB(N), XIN(IN), YIN(IN), UIN(IN)
С
С
  Read the names and open the input and output files
С
     WRITE (*,'(A)')' Name of the INPUTFILE (max.15 characters)'
     READ
           (*,'(A)') INPUTFILE
     WRITE (*,'(A)')' Name of the OUTPUTFILE (max.15 characters)'
     READ (*, '(A) ') OUTPUTFILE
     OPEN (UNIT=1, FILE=INPUTFILE)
     OPEN (UNIT=2, FILE=OUTPUTFILE)
C
С
  Read data from INPUTFILE
С
      CALL INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN, NL, NB)
С
C
   Compute the G matrix
С
      CALL GMATR (XL, YL, XM, YM, G, N, NL, NB)
С
С
   Compute the H matrix
С
      CALL HMATR (XL, YL, XM, YM, H, N, NL, NB)
С
С
  Form the system of equations AX=B
C
      CALL ABMATR (G, H, A, UNB, UB, INDEX, N)
С
  Solve the system of equations
C
С
      CALL SOLVEQ(A, UNB, N, LSING)
С
   Form the vectors U and UN of all the boundary values
С
С
      CALL REORDER (UB, UNB, INDEX, N)
```

```
С
С
  Compute the values of U at the internal points
С
      CALL UINTER (XL, YL, XIN, YIN, UB, UNB, UIN, N, IN, NL, NB)
С
  Print the results in the OUTPUTFILE
С
С
      CALL OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, N, IN)
С
С
   Close input and output files
С
      CLOSE(1)
      CLOSE(2)
С
      STOP
      END
С
С
        Cesses
С
      SUBROUTINE INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN, NL, NB)
С
  This subroutine reads the input data from the input file
Ċ
  and writes them in the output file
C
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*80 NAME, TITLE
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), INDEX(N), UB(N), NL(NB)
C
      WRITE(2,100)
  100 FORMAT('',69('*'))
С
C
  Read user's name
С
      READ(1, '(A) ')NAME
С
      WRITE(2, '(A) ')NAME
C
  Read the title of the program
С
С
      READ(1, '(A)')TITLE
С
      WRITE(2, '(A) ')TITLE
С
   Read the number of the last element on each boundary
С
C
      READ(1,*)(NL(I),I=1,NB)
С
      WRITE(2,200)N, IN, NB
  200 FORMAT (//'BASIC PARAMETERS'//2X, 'NUMBER OF BOUNDARY ELEMENTS='
     1,13/2X, 'NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED
     1=',13//2X,'NUMBER OF BOUNDARIES =',13)
      DO 250 I=1,NB
  250 WRITE(2,260) I, NL(I)
  260 FORMAT(2X, 'BOUNDARY', I3, 2X, 'LAST BOUNDARY ELEMENT=', I3)
С
   Read the coordinates XL, YL of the extreme points of the boundary elements
С
С
      READ(1, *) (XL(I), YL(I), I=1, N)
С
   Write the coordinates in the output file
С
С
      WRITE(2,300)
  300 FORMAT(//2X, 'COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY '
     l'ELEMENTS',//2X,'POINT',9X,'XL',15X,'YL')
```

```
DO 20 I=1,N
   20 WRITE(2,400) I,XL(I),YL(I)
  400 FORMAT(2X,13,2(3X,E14.5))
С
C
   Read the boundary conditions and store them in UB(I).
   If INDEX(I)=0, the value UB(I) is the prescribed value of u.
С
C If INDEX(I) =1, the normal derivative un is prescribed.
С
      READ(1,*) (INDEX(I), UB(I), I=1, N)
C
С
   Write the boundary conditions in the output file
С
      WRITE(2,500)
  500 FORMAT(//2X, 'BOUNDARY CONDITIONS'//2X, 'NODE', 6X, 'INDEX',
     1 7X, 'PRESCRIBED VALUE')
      DO 30 I=1,N
   30 WRITE(2,600) I, INDEX(I), UB(I)
  600 FORMAT (2X, I3, 9X, I1, 8X, E14.5)
С
C
  Read the coordinates of the internal points
С
      READ(1, *) (XIN(I), YIN(I), I=1, IN)
      RETURN
      END
C
С
С
      SUBROUTINE GMATE (XL, YL, XM, YM, G, N, NL, NB)
С
  This subroutine computes the elements of the G matrix
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), NL(NB)
      DIMENSION G(N,N)
C
С
  Compute the nodal coordinates and store them in the arrays XM
С
   and YM
С
      IF (NB.GT.1) GOTO 5
      XL(NL(1)+1) = XL(1)
      YL(NL(1)+1) = YL(1)
С
    5 DO 10 I=1,N
      XM(I) = (XL(I) + XL(I+1))/2.
   10 YM(I) = (YL(I) + YL(I+1))/2.
      IF(NB.LE.1) GO TO 40
      XM(NL(1)) = (XL(NL(1)) + XL(1)) / 2.
      YM(NL(1)) = (YL(NL(1)) + YL(1))/2.
      DO 15 K=2, NB
      XM(NL(K)) = (XL(NL(K)) + XL(NL(K-1) + 1))/2.
   15 YM(NL(K)) = (YL(NL(K)) + YL(NL(K-1) + 1))/2.
С
С
   Compute the elements of matrix G
C
   40 DO 20 I=1,N
      XO = XM(I)
      Y0 × YM(I)
      DO 20 J=1,N
      X1=XL(J)
      Y1 = YL(J)
      X2 \neq XL(J+1)
      Y2 = YL(J+1)
      IF(NB.LE.1)GOTO 60
        IF(J.NE.NL(1))GOTO 50
```

```
X2 = XL(1)
          Y2=YL(1)
          GOTO 60
   50
        DO 30 K=2,NB
        IF(J.NE.NL(K))GOTO 30
          X2 = XL(NL(K-1)+1)
          Y2 = YL(NL(K-1)+1)
          GOTO 60
        CONTINUE
   30
   60 IF(I.NE.J)THEN
        CALL RLINTC (X0, Y0, X1, Y1, X2, Y2, RESULT)
        G(I,J)=RESULT
      ELSEIF(I.EQ.J) THEN
        CALL SLINTC(X1,Y1,X2,Y2,RESULT)
        G(I,J)=RESULT
      ENDIF
   20 CONTINUE
      RETURN
      END
С
С
C=
          C
      SUBROUTINE HMATR(XL,YL,XM,YM,H,N,NL,NB)
С
C
   This subroutine computes the elements of the H matrix
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), NL(NB)
      DIMENSION H(N,N)
C
      PI=ACOS(-1.)
с
С
  Compute the nodal coordinates and store them in the arrays XM
С
  and YM
С
      IF (NB.GT.1) GOTO 5
      XL(NL(1)+1) = XL(1)
      YL(NL(1)+1) = YL(1)
С
    5 DO 10 I=1,N
      XM(I) = (XL(I) + XL(I+1))/2.
   10 YM(I) = (YL(I) + YL(I+1))/2.
      IF(NB.LE.1) GO TO 40
      XM(NL(1)) = (XL(NL(1)) + XL(1))/2.
      YM(NL(1)) = (YL(NL(1)) + YL(1))/2.
      DO 15 K=2,NB
      XM(NL(K)) = (XL(NL(K)) + XL(NL(K-1) + 1))/2.
   15 YM(NL(K)) = (YL(NL(K)) + YL(NL(K-1)+1))/2.
С
С
   Compute the elements of H matrix
C
   40 DO 20 I=1,N
      X0 = XM(I)
      Y0 = YM(I)
      DO 20 J=1,N
      X1 = XL(J)
      Y1=YL(J)
      X2 = XL(J+1)
      Y2=YL(J+1)
      IF(NB.LE.1)GOTO 60
        IF(J.NE.NL(1))GOTO 50
          X2 = XL(1)
          Y2 = YL(1)
          GOTO 60
```

```
DO 30 K=2,NB
   50
        IF(J.NE.NL(K))GOTO 30
         X2 = XL(NL(K-1)+1)
          Y2 = YL(NL(K-1)+1)
          GOTO 60
   30
       CONTINUE
   60 IF (I.NE.J) THEN
       CALL DALPHA (X0,Y0,X1,Y1,X2,Y2,RESULT)
       H(I,J)=RESULT
      ELSEIF (I.EQ.J) THEN
       H(I, J) = -0.5
      ENDIF
   20 CONTINUE
     RETURN
      END
с
C
C=:
      С
     SUBROUTINE UINTER (XL, YL, XIN, YIN, UB, UNB, UIN, N, IN, NL, NB)
С
С
  This subroutine computes the values of u at the internal points
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), UB(N), UNB(N), UIN(IN)
     DIMENSION NL(NB)
С
с
  Compute the values of u at the internal points
С
     IF (NB.GT.1) GOTO 5
     XL(NL(1)+1) = XL(1)
     YL(NL(1)+1) = YL(1)
С
  5 DO 10 KK=1, IN
     UIN(KK) = 0.
     DO 20 J=1,N
     X0=XIN(KK)
     Y0=YIN(KK)
     X1 = XL(J)
     Y1=YL(J)
     X2 = XL(J+1)
     Y2 = YL(J+1)
     IF (NB.LE.1) GOTO 60
       IF(J.NE.NL(1))GOTO 50
         X2 = XL(1)
         Y2 = YL(1)
         GOTO 60
  50
       DO 30 K=2,NB
       IF(J.NE.NL(K))GOTO 30
         X2 = XL(NL(K-1)+1)
         Y2 = YL(NL(K-1)+1)
         GOTO 60
  30
       CONTINUE
  60 CALL DALPHA (X0, Y0, X1, Y1, X2, Y2, RESH)
     CALL RLINTC (X0, Y0, X1, Y1, X2, Y2, RESG)
  20 UIN(KK)=UIN(KK)+RESH*UB(J)-RESG*UNB(J)
  10 CONTINUE
     RETURN
     END
С
С
C====
```

Example 4.3

This example demonstrates program LABECONMU for the solution of a simple potential problem with mixed boundary conditions. The square domain Ω is doubly connected, namely it contains a hole. Its outer boundary has been discretized into 16 constant elements, while the inner one into 8 elements. The solution is sought at 8 internal points. The data and the boundary discretization are shown in Figs. 4.17 and 4.18. The data file has been created using program RECT-2.FOR.

The exact solution is: u(x,y) = 100(1+x).



Figure 4.17 Doubly connected domain Ω and boundary conditions.

EXAMPLE 4.3 (DATA)

Example 4.3 16 24 .00000000 .00000000 .25000000 .00000000 .50000000 .00000000 .75000000 .00000000 .00000000 1.00000000 1.00000000 .25000000 1.00000000 .50000000 1.00000000 .75000000 1.00000000 1.00000000 .75000000 1.00000000

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Figure 4.18 Boundary element discretization for the doubly connected domain of Example 4.3.

	.50000000	1.00000000
	.25000000	1.00000000
	.00000000	1.00000000
	.00000000	.75000000
	.00000000	.50000000
	.00000000	.25000000
	.30000000	.3000000
	.30000000	.50000000
	.30000000	.7000000
	.50000000	.7000000
	.70000000	.7000000
	.70000000	.50000000
	.70000000	.3000000
	.50000000	.3000000
1		.0000000
1		.00000000
1		.00000000
1		.00000000
0	200	.0000000
0	200	.0000000
0	200	.00000000
0	200	.0000000
1		.00000000
1		.00000000
1		.00000000
1		.0000000
0	100	.0000000
0	100	.00000000
0	100	.00000000
0	100	.00000000
0	130	.0000000
0	130	.0000000
1		.00000000
1		.00000000
1	-100	.0000000
1	-100	.0000000
1		.00000000
1		.0000000
.15000000	.15000000	
-----------	-----------	
.50000000	.15000000	
.85000000	.15000000	
.15000000	.50000000	
.85000000	.50000000	
.15000000	.85000000	
.50000000	.85000000	
.85000000	.85000000	

EXAMPLE 4.3 (RESULTS)

BASIC PARAMETERS

NUMBER OF BOUNDARY ELEMENTS= 24 NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED= 8

NUMBER OF BOUNDARIES = 2 BOUNDARY 1 LAST BOUNDARY ELEMENT= 16 BOUNDARY 2 LAST BOUNDARY ELEMENT= 24

COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELEMENTS

POINT	XL	YL
1	.00000E+00	.00000E+00
2	.25000E+00	.00000E+00
3	.50000E+00	.00000E+00
4	.75000E+00	.00000E+00
5	.10000E+01	.00000E+00
6	.10000E+01	.25000E+00
7	.10000E+01	.50000E+00
8	.10000E+01	.75000E+00
9	.10000E+01	.10000E+01
10	.75000E+00	.10000E+01
11	.50000E+00	.10000E+01
12	.25000E+00	.10000E+01
13	.00000E+00	.10000E+01
14	.00000E+00	.75000E+00
15	.00000E+00	.50000E+00
16	.00000E+00	.25000E+00
17	.30000E+00	.30000E+00
18	.30000E+00	.50000E+00
19	.30000E+00	.70000E+00
20	.50000E+00	.70000E+00
21	.70000E+00	.70000E+00
22	.70000E+00	.50000E+00
23	.70000E+00	.30000E+00
24	.50000E+00	.30000E+00

BOUNDARY CONDITIONS

NODE	INDEX	PRESCRIBED VALUE
1	1	.00000E+00
2	1	.00000E+00

3	1	.00000E+00
4	1	.00000E+00
5	0	.20000E+03
6	0	.20000E+03
7	0	.20000E+03
8	0	.20000E+03
9	1	.00000E+00
10	1	.00000E+00
11	1	.00000E+00
12	1	.00000E+00
13	0	.10000E+03
14	0	.10000E+03
15	0	.10000E+03
16	0	.10000E+03
17	0	.13000E+03
18	0	.13000E+03
19	1	.00000E+00
20	1	.00000E+00
21	1	10000E+03
22	1	10000E+03
23	1	.00000E+00
24	1	.00000E+00

The system has been solved regularly

RESULTS

BOUNDARY NODES

х	Y	U	Un
.12500E+00	.00000E+00	.11212E+03	.00000E+00
.37500E+00	.00000E+00	.13776E+03	.00000E+00
.62500E+00	.00000E+00	.16273E+03	.00000E+00
.87500E+00	.00000E+00	.18811E+03	.00000E+00
.10000E+01	.12500E+00	.20000E+03	.10536E+03
.10000E+01	.37500E+00	.20000E+03	.95220E+02
.10000E+01	.62500E+00	.20000E+03	.95220E+02
.10000E+01	.87500E+00	.20000E+03	.10536E+03
.87500E+00	.10000E+01	.18811E+03	.00000E+00
.62500E+00	.10000E+01	.16273E+03	.00000E+00
.37500E+00	.10000E+01	.13776E+03	.00000E+00
.12500E+00	.10000E+01	.11212E+03	.00000E+00
.00000E+00	.87500E+00	.10000E+03	10777E+03
.00000E+00	.62500E+00	.10000E+03	99560E+02
.00000E+00	.37500E+00	.10000E+03	99560E+02
.00000E+00	.12500E+00	.10000E+03	10777E+03
.30000E+00	.40000E+00	.13000E+03	.10833E+03
.30000E+00	.60000E+00	.13000E+03	.10833E+03
.40000E+00	.70000E+00	.14054E+03	.00000E+00
.60000E+00	.70000E+00	.16002E+03	.00000E+00
.70000E+00	.60000E+00	.17168E+03	10000E+03
.70000E+00	.40000E+00	.17168E+03	10000E+03
.60000E+00	.30000E+00	.16002E+03	.00000E+00
.40000E+00	.30000E+00	.14054E+03	.00000E+00

INTERNAL POINTS

x	Y	SOLUTION U	
.15000E+00	.15000E+00	.11508E+03	
.50000E+00	.15000E+00	.15026E+03	
.85000E+00	.15000E+00	.18525E+03	
.15000E+00	.50000E+00	.11507E+03	
.85000E+00	.50000E+00	.18569E+03	
.15000E+00	.85000E+00	.11508E+03	
.50000E+00	.85000E+00	.15026E+03	
.85000E+00	.85000E+00	.18525E+03	

4.9 The method of subdomains

In certain problems the material properties of the body are piecewise continuous. Such, for example, is the torsion of a composite bar consisting of two or more materials of different shear moduli. Another example is the problem of heat conduction in a body having different coefficients of thermal conductivity in two or more subregions. In the literature such a body is referred to as *multi-zone body* and the domain it occupies a *composite domain*.

Potential problems in composite domains can be solved by applying the BEM separately to each of its subdomains. The reason is that the fundamental solution is valid only for homogeneous domains. Next, without restricting the generality, we will apply the BEM to the composite domain of Fig. 4.19, which consists of the three subdomains Ω_1 , Ω_2 and Ω_3 . In discretizing the boundaries of these subdomains, special care is taken to have the same discretization on either side of each interface, namely, on common parts of the boundaries between subdomains (see Fig. 4.20).



Figure 4.19 Composite domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.



Figure 4.20 Boundary element discretization of composite domain $(\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$. Subdomains and interfaces.

In each subdomain the following vectors are defined:

Subdomain Ω_1

$\{u\}_1^1, \ \{u_n\}_1^1$	Nodal values on part Γ_1 of the external boundary
$\{u\}_{12}^1, \ \{u_n\}_{12}^1$	Nodal values on the interface Γ_{12}
$\{u\}_{13}^1,\;\{u_n\}_{13}^1$	Nodal values on the interface Γ_{13}

In the foregoing notation, the superscript denotes the subdomain, while the subscript denotes the neighboring subdomains or the corresponding interface. The number of the nodal points on Γ_1 , Γ_{12} and Γ_{13} are N_1 , N_{12} and N_{13} , respectively.

Subdomain Ω_2

$\{u\}_2^2$,	$\{u_n\}_2^2$	Nodal	values	on part	Γ_2	of the	external	boundary
---------------	---------------	-------	--------	---------	------------	--------	----------	----------

- $\{u\}_{12}^2, \ \{u_n\}_{12}^2$ Nodal values on the interface Γ_{12}
- $\{u\}_{23}^2$, $\{u_n\}_{23}^2$ Nodal values on the interface Γ_{23}

The number of the nodal points on Γ_2 , Γ_{12} and Γ_{23} are denoted by N_2 , N_{12} and N_{23} , respectively.

Subdomain Ω_3

 $\{u\}_3^3, \{u_n\}_3^3$ Nodal values on part Γ_3 of the external boundary

 $\{u\}_{13}^3, \{u_n\}_{13}^3$ Nodal values on the interface Γ_{13}

 $\{u\}_{23}^3$, $\{u_n\}_{23}^3$ Nodal values on the interface Γ_{23}

The number of the nodal points on Γ_3 , Γ_{13} and Γ_{23} are N_3 , N_{13} and N_{23} , respectively.

The boundary conditions are specified only on the external boundary Γ of Ω , that is only on the parts Γ_1 , Γ_2 , Γ_3 ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$). Both quantities u and u_n are unknown on either side of the interfaces Γ_{12} , Γ_{13} , Γ_{23} . Therefore, the total number of boundary unknowns in each subdomain is:

Subdomain Ω_1 :	N_1 on Γ_1 , $2N_{12}$ on Γ_{12} , $2N_{13}$ on Γ_{13} , total $N_1 + 2N_{12} + 2N_{13}$
Subdomain Ω_2 :	N_2 on Γ_2 , $2N_{12}$ on Γ_{12} , $2N_{23}$ on Γ_{23} , total $N_2 + 2N_{12} + 2N_{23}$
Subdomain Ω_3 :	N_3 on Γ_3 , $2N_{13}$ on Γ_{13} , $2N_{23}$ on Γ_{23} , total $N_3 + 2N_{13} + 2N_{23}$

The available, however, equations for the evaluation of the unknown boundary quantities are:

from subdomain Ω_1 : $N_1 + N_{12} + N_{13}$ from subdomain Ω_2 : $N_2 + N_{12} + N_{23}$ from subdomain Ω_3 : $N_3 + N_{13} + N_{23}$

Therefore, the number of unknowns exceeds the number of available equations by $2(N_{12} + N_{13} + N_{23})$. The additional equations result from physical considerations, which are the so-called *continuity conditions* at the interfaces. These conditions express:

(a) *Continuity of the potential.* The values of the potential on each side of the interface separating two subdomains are equal,

${u}_{12}^1 = {u}_{12}^2$		
$\{u\}_{13}^1 = \{u\}_{13}^3$	}	(4.49)
${u}_{23}^2 = {u}_{23}^3$	J	

(b) Continuity of the flux. The flux q_n is a quantity related to the physical problem described by the potential equation. For example, in heat conduction problems the flux is given by Fourier's law $q_n = k u_n$, where k is the coefficient of thermal conductivity of the material. For the torsion problem the expression for the flux is more complicated. The outcoming flux from one subdomain is equal to the incoming flux in the adjacent subdomain. Thus, if q_n denotes the flux along the normal to the interface, its continuity across the interface requires

$$\begin{cases}
q_n\}_{12}^2 = -\{q_n\}_{12}^1 \\
\{q_n\}_{13}^3 = -\{q_n\}_{13}^1 \\
\{q_n\}_{23}^3 = -\{q_n\}_{23}^2
\end{cases}$$
(4.50)

The minus sign in the right hand side of Eqs. (4.50) is explained by the fact that the positive direction for the flux coincides with the outward normal vector \mathbf{n} . This means that the two flux vectors at the common interface of adjacent subdomains are of opposite directions, because their corresponding normal vectors are opposite too (Fig. 4.20). Generally, the formulation of the continuity condition for the flux requires familiarity with the physical problem under consideration. For this reason, the above concepts will become more clear after studying Chapter 6.

~

Let us apply the flux continuity conditions to the heat flow problem. If the coefficients of thermal conductivity in the subdomains Ω_1 , Ω_2 and Ω_3 are denoted by k_1 , k_2 and k_3 , respectively, then Eqs. (4.50) may be written as

$$\{u_{n}\}_{12}^{2} = -[k_{12}]\{u_{n}\}_{12}^{1}$$

$$\{u_{n}\}_{13}^{3} = -[k_{13}]\{u_{n}\}_{13}^{1}$$

$$\{u_{n}\}_{23}^{3} = -[k_{23}]\{u_{n}\}_{23}^{2}$$

$$(4.51)$$

where $[k_{12}]$, $[k_{13}]$, $[k_{23}]$ are square diagonal matrices with elements $k_{12} = k_1/k_2$, $k_{13} = k_1/k_3$ and $k_{23} = k_2/k_3$, respectively.

Equations (4.49) and (4.51) provide now the required $2(N_{12} + N_{13} + N_{23})$ additional equations for establishing all the unknowns of the problem.

On the basis of the foregoing, the matrix equations for the boundary of each subdomain become:

(i) For the boundary of subdomain Ω_1

$$[H]^{1}\{u\}^{1} = [G]^{1}\{u_{n}\}^{1}$$

or

$$\begin{bmatrix} [H]_{1}^{1} & [H]_{12}^{1} & [H]_{13}^{1} \end{bmatrix} \begin{cases} \{u\}_{1}^{1} \\ \{u\}_{12}^{1} \\ \{u\}_{13}^{1} \end{cases} = \begin{bmatrix} [G]_{1}^{1} & [G]_{12}^{1} & [G]_{13}^{1} \end{bmatrix} \begin{cases} \{u_{n}\}_{1}^{1} \\ \{u_{n}\}_{12}^{1} \\ \{u_{n}\}_{13}^{1} \end{cases}$$

The boundary conditions on part Γ_1 of the external boundary influence only the first term in each side of the above equation. Incorporating these conditions, we get

$$\begin{bmatrix} [A]_{1}^{1} & [H]_{12}^{1} & [H]_{13}^{1} \end{bmatrix} \begin{cases} \{x\}_{1}^{1} \\ \{u\}_{12}^{1} \\ \{u\}_{13}^{1} \end{bmatrix} = \{B\}_{1}^{1} + \begin{bmatrix} [G]_{12}^{1} & [G]_{13}^{1} \end{bmatrix} \begin{bmatrix} \{u_{n}\}_{12}^{1} \\ \{u_{n}\}_{13}^{1} \end{bmatrix}$$
(4.52)

The vector $\{x\}_1^1$ contains all the unknown boundary quantities of Γ_1 . The matrix $[A]_1^1$ and the vector $\{B\}_1^1$ are derived using the same procedure as for the case of Eq. (4.9).

(ii) For the boundary of subdomain Ω_2

$$[H]^{2} \{u\}^{2} = [G]^{2} \{u_{n}\}^{2}$$

or

$$\begin{bmatrix} [H]_2^2 & [H]_{12}^2 & [H]_{23}^2 \end{bmatrix} \begin{cases} \{u\}_2^2 \\ \{u\}_{12}^2 \\ \{u\}_{23}^2 \end{cases} = \begin{bmatrix} [G]_2^2 & [G]_{12}^2 & [G]_{23}^2 \end{bmatrix} \begin{cases} \{u_n\}_2^2 \\ \{u_n\}_{12}^2 \\ \{u_n\}_{23}^2 \end{cases}$$

Using the continuity conditions (4.49) and (4.51), the above equation yields

$$\begin{bmatrix} [H]_2^2 & [H]_{12}^2 & [H]_{23}^2 \end{bmatrix} \begin{cases} \{u\}_2^2 \\ \{u\}_{12}^1 \\ \{u\}_{23}^2 \end{cases} = \begin{bmatrix} [G]_2^2 & -[G]_{12}^2 [k_{12}] & [G]_{23}^2 \end{bmatrix} \begin{cases} \{u_n\}_2^2 \\ \{u_n\}_{12}^1 \\ \{u_n\}_{23}^2 \end{cases}$$

Subsequently, applying the boundary conditions from part Γ_2 of the external boundary, we arrive at

$$\begin{bmatrix} [A]_{2}^{2} & [H]_{12}^{2} & [H]_{23}^{2} \end{bmatrix} \begin{cases} \{x\}_{2}^{2} \\ \{u\}_{12}^{1} \\ \{u\}_{23}^{2} \end{cases} = \{B\}_{2}^{2} + \begin{bmatrix} -[G]_{12}^{2}[k_{12}] & [G]_{23}^{2} \end{bmatrix} \begin{cases} \{u_{n}\}_{12}^{1} \\ \{u_{n}\}_{23}^{2} \end{cases}$$
(4.53)

(iii) For the boundary of subdomain Ω_3

 $[H]^3 \{u\}^3 = [G]^3 \{u_n\}^3$

or

$$\begin{bmatrix} [H]_3^3 & [H]_{13}^3 & [H]_{23}^3 \end{bmatrix} \begin{cases} \{u\}_3^3 \\ \{u\}_{13}^3 \\ \{u\}_{23}^3 \end{cases} = \begin{bmatrix} [G]_3^3 & [G]_{13}^3 & [G]_{23}^3 \end{bmatrix} \begin{cases} \{u_n\}_3^3 \\ \{u_n\}_{13}^3 \\ \{u_n\}_{23}^3 \end{cases}$$

$$\begin{bmatrix} [H]_{3}^{3} & [H]_{13}^{3} & [H]_{23}^{3} \end{bmatrix} \begin{cases} \{u\}_{3}^{3} \\ \{u\}_{13}^{1} \\ \{u\}_{23}^{2} \end{cases} = \begin{bmatrix} [G]_{3}^{3} & -[G]_{13}^{3} [k_{13}] & -[G]_{23}^{3} [k_{23}] \end{bmatrix} \begin{cases} \{u_{n}\}_{3}^{3} \\ \{u_{n}\}_{13}^{1} \\ \{u_{n}\}_{23}^{2} \end{cases}$$

Finally, incorporating the boundary conditions for part Γ_3 of the external boundary, the last equation gives

$$\begin{bmatrix} [A]_{3}^{3} & [H]_{13}^{3} & [H]_{23}^{3} \end{bmatrix} \begin{cases} \{x\}_{3}^{3} \\ \{u\}_{13}^{1} \\ \{u\}_{23}^{2} \end{cases} = \{B\}_{3}^{3} + \begin{bmatrix} -[G]_{13}^{3}[k_{13}] & -[G]_{23}^{3}[k_{23}] \end{bmatrix} \begin{cases} \{u_{n}\}_{13}^{1} \\ \{u_{n}\}_{23}^{2} \end{cases}$$

$$(4.54)$$

Equations (4.52), (4.53) and (4.54) of the three subdomains may then be combined in a single matrix equation as

$$[A]\{X\} = \{B\}$$
(4.55)

where

 $\{X\}$: Vector consisting of all the unknown values on the external boundary and on the interfaces. Its dimension is:

 $N = N_1 + N_2 + N_3 + 2N_{12} + 2N_{13} + 2N_{23}$

[A]: Known square coefficient matrix of dimensions $N \times N$

 $\{B\}$: Known vector of dimension N

The vectors $\{X\}$, $\{B\}$ and the matrix [A] are defined by the equations

$$\{X\} = \begin{cases} \{x\}_{1}^{1} \\ \{x\}_{2}^{2} \\ \{x\}_{3}^{3} \\ \{u\}_{12}^{1} \\ \{u\}_{13}^{1} \\ \{u\}_{23}^{2} \\ \{u_{n}\}_{12}^{1} \\ \{u_{n}\}_{12}^{1} \\ \{u_{n}\}_{13}^{1} \\ \{u_{n}\}_{23}^{2} \end{cases}, \quad \{B\} = \begin{cases} \{B\}_{1}^{1} \\ \{B\}_{2}^{2} \\ \{B\}_{3}^{3} \end{cases}$$

$$(4.56)$$

	$[A]_{1}^{1}$	[0]	[0]	$[H]_{12}^1$	$[H]^{1}_{13}$	[0]	$-[G]_{12}^1$	$-[G]_{13}^1$	[0]	
[A] =	[0]	$[A]_{2}^{2}$	[0]	$[H]_{12}^2$	[0]	$[H]_{23}^2$	$[G]_{12}^2[k_{12}]$	[0]	$-[G]_{23}^2$	
	[0]	[0]	$[A]_3^3$	[0]	$[H]^3_{13}$	$[H]_{23}^3$	[O]	$[G]^3_{13}[k_{13}]$	$[G]^3_{23}[k_{23}]$	
									(4.5	7)

We notice that matrix [A] is not fully populated. Its structure, therefore, allows the use of special techniques for the solution of Eq. (4.55), which reduce running time and disk space requirements. The method of subdomains may also be employed for long and slender homogeneous domains to overcome numerical problems associated with the integration of the fundamental solution over long distances. By splitting the domain into two or more subdomains (see Fig. 4.21), the aspect ratio of each subdomain is reduced, and the influence matrices [H] and [G] are computed more accurately.



Figure 4.21 Long and slender homogeneous domain divided in three subdomains.

4.10 References

In the present chapter the BEM was presented as a numerical method suitable for solving problems described by the potential equation. The employed boundary discretization was the simplest one, that of constant elements. For further reading on the material of this chapter, one should look in the books by Jaswon and Symm [14], Brebbia [15], Banerjee and Butterfield [16], Brebbia and Dominguez [17] and Kane [18]. The computer language employed for the programs was FORTRAN 77 of the Microsoft Fortran PowerStation (Professional). The FORTRAN 90, though more powerful, was not selected, because many readers of this book might not be familiar with it yet.

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Problems

- **4.1.** Write in FORTRAN a subroutine that computes the derivatives $u_{,x}$ and $u_{,y}$ at internal points $P \in \Omega$.
- **4.2.** Modify appropriately the program LABECON so that it can be used to solve the anisotropic problem.
- **4.3.** For the composite domain of the following figure and the indicated discretization, compute the matrix [A] and the vector $\{B\}$ of Eq. (4.55).



Figure P4.3

Boundary Element Technology

5.1 Introduction

In the previous chapter, the boundary value problem described by the Poisson's differential equation (3.6a) and the boundary condition (3.6b) was modeled numerically by the BEM. This formulation originated from the discretization of the boundary integral equation (Eq. 3.37) and resulted in the system of linear algebraic equations (4.9), which approximates the solution of the integral equation. The accuracy of the approximation and the efficiency of the BEM depends on the boundary discretization technique (i.e., the type of employed element), and on the method used for the integration of the kernel functions over the elements.

The constant element was presented in Chapter 4. This element approximates the actual geometry by a straight line, while the unknown boundary quantity is assumed to be constant on the element, resulting in a discontinuous distribution on the boundary. A better approximation of the boundary quantity can be achieved by adopting linear variation over the element. Even the linear element is not an ideal one as it can not approximate accurately a curved boundary. For this reason, higher order elements have been developed, namely, elements that approximate both the boundary geometry and the boundary quantities by higher order interpolation polynomials—usually of second or third degree. Evidently, these elements model more accurately the curved boundaries and the distribution of the boundary quantities on the element. However, they have the weak point that the functions being integrated over each element become more complicated and the computer time is considerably increased.

Generally, the boundary elements may be classified in the following three categories [1]:

(a) Subparametric elements. The polynomial approximating the geometry of the boundary is of a lower degree than that approximating the variation of the

boundary quantity, e.g. the element is a straight line, whereas the boundary quantity varies parabolically.

- (b) *Isoparametric elements*. The geometry and the boundary quantity are approximated by the same degree polynomials, e.g. the linear and the parabolic elements defined in Chapter 4 are isoparametric.
- (c) Superparametric elements. The geometry is approximated by a higher degree polynomial than that approximating the boundary quantity, e.g. the element modeled by a parabolic arc, whereas the boundary quantity is constant or varies linearly on it.



(b) Discontinuous element

Figure 5.1 Continuous and discontinuous linear elements.

The use of the subparametric and superparametric elements is quite limited. Subparametric elements, however, result inevitably as a degenerate case of parabolic elements for rectilinear boundaries. Superparametric elements, in which the boundary geometry is modeled by a parabolic arc and the boundary quantity is constant, have been introduced and widely used by Katsikadelis and his co-workers [2–6]. This element has the advantages of the constant element regarding the simplicity of the assembly procedure for the influence matrices, whereas it approximates the curvilinear boundary with great accuracy. In general, the isoparametric elements are the most widely used elements, especially in commercial BEM codes.

The boundary elements are also distinguished in *continuous* and *discontinuous*. Continuous elements have nodes at their extreme points, and therefore they share nodes with the adjacent elements, while discontinuous elements have nodes located away from the extreme points. In the sections that follow, we will discuss the linear and the parabolic elements for continuous and discontinuous modeling.

5.2 Linear elements

As it was mentioned before, linear elements approximate the geometry of the boundary by straight lines and the boundary quantity by a linear function on each element. In order to establish the expression for the variation of the boundary quantity over an element, its values at two nodal points are required. For this purpose, it is convenient to introduce a local coordinate system Ox'y' on each element, where $-\ell/2 \le x' \le \ell/2$, ℓ being the length of the element. For continuous elements the nodal points are placed at the extreme points (Fig. 5.1a), whereas for discontinuous elements at points between the end points (Fig. 5.1b).



Figure 5.2 Elliptic boundary modeled by 12 continuous elements.

Figure 5.2 shows the discretization of an elliptic boundary into 12 continuous linear elements, whereas Fig. 5.3 shows the discretization of the same boundary into 12 discontinuous linear elements. From these two figures, it becomes evident that the discretization of the boundary using discontinuous elements requires twice as many nodes compared to the continuous elements. However, a better approximation is achieved by employing the discontinuous elements (see Fig. 5.4). Anyhow, the advantage of the discontinuous element is not the improved accuracy as compared to the continuous element, but rather its capability to overcome computational problems arising at points where the boundary quantity is discontinuous, e.g. the normal derivative at corner points. *Hybrid* or *continuous–discontinuous* elements have also been invented, that is, elements having only one of the nodes placed at an extreme point.



Figure 5.3 Elliptic boundary modeled by 12 discontinuous elements.

Let us consider the j-th element of the discretized boundary having end points j and j+1, and length ℓ_j (Fig. 5.5). In the local system of axes O'x'y' its geometry is described by the equations

$$x' = x' \qquad \left(-\ell_{j}/2 \le x' \le \ell_{j}/2\right)$$
$$y' = 0$$

whereas in the global system of axes Oxy by the equations



(b) Discontinuous elements





Figure 5.5 Global (Oxy) and local (O'x'y') systems of axes for the j-th element.

$$x = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{\ell_j} x'$$
(5.1a)

$$y = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{\ell_j} x'$$
(5.1b)

Recall that the positive direction of x' on the element is from point j towards point j+1, since the nodes and extreme points of the discretized boundary are numbered in the counter-clockwise sense. The interval $[-\ell_j/2, \ell_j/2]$ is normalized by setting

$$\xi = \frac{x'}{\ell_j/2} \tag{5.2}$$

and then Eqs. (5.1) become

$$x = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{2} \xi$$
(5.3a)

$$y = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{2} \xi$$
(5.3b)

where $-1 \leq \xi \leq 1$.

Equations (5.3) may be rewritten as

$$x(\xi) = \frac{1}{2}(1-\xi)x_j + \frac{1}{2}(1+\xi)x_{j+1}$$
(5.4a)

$$y(\xi) = \frac{1}{2}(1-\xi)y_j + \frac{1}{2}(1+\xi)y_{j+1}$$
(5.4b)

The variation of the boundary quantity u (or $u_n = \partial u / \partial n$) is linear on the element. Hence, its distribution in the local system of axes is given by the expression

$$u = \frac{u^{j+1} + u^{j}}{2} + \frac{u^{j+1} - u^{j}}{\ell_{j}} x'$$

or

$$u = \frac{u^{j+1} + u^{j}}{2} + \frac{u^{j+1} - u^{j}}{2} \xi$$

or

$$u(\xi) = \frac{1}{2}(1-\xi)u^{j} + \frac{1}{2}(1+\xi)u^{j+1}$$
(5.5)

Using local numbering for the nodes where j and j+1 are renamed to 1 and 2, respectively, Eqs. (5.4a), (5.4b) and (5.5) may all be expressed through the following general equation

$$f(\xi) = \psi_1(\xi) f_1 + \psi_2(\xi) f_2$$
(5.6)

where f_1 and f_2 are the values of the function f(x) at the nodes 1 and 2, and $f(\xi)$ represents any of the functions $x(\xi)$, $y(\xi)$, $u(\xi)$ or $u_n(\xi)$. The functions $\psi_1(\xi)$ and $\psi_2(\xi)$ are given as

$$\psi_1(\xi) = \frac{1}{2}(1-\xi) \tag{5.7a}$$

$$\psi_2(\xi) = \frac{1}{2}(1+\xi) \tag{5.7b}$$

and they express the influence of the nodal values f_j and f_{j+1} on the expression of $f(\xi)$ for the linear element. They are the functions of the linear interpolation and they are referred to as *linear shape functions*. From the above, one readily concludes that the linear element is isoparametric.

5.3 The BEM with linear boundary elements

In this section, we will present the BEM for the potential equation using continuous linear elements. The number of the elements is equal to the number of nodes. The actual boundary is, thus, modeled by an inscribed polygon with the nodes placed at corner points.

In the case of constant elements, the substitute boundary which models the actual one, is always smooth at the nodes and, therefore, the integral equation (3.29) is employed. In the case of linear elements, however, the nodes lie at the corners of the polygon and, consequently, the integral equation (3.28) should be applied instead with $\alpha = \alpha^i$, where α^i is the angle between the elements (i-1) and i (Fig. 5.6). Thus, after discretizing the boundary into N linear elements, Eq. (3.28) becomes

$$\varepsilon^{i} u^{i} = -\sum_{j=1}^{N} \int_{\Gamma_{j}} v \, u_{n} \, ds + \sum_{j=1}^{N} \int_{\Gamma_{j}} u \, v_{n} \, ds \tag{5.8}$$

where $\varepsilon^i = \alpha^i/2\pi$ and $v_n = \partial v/\partial n$ denotes the derivative of the fundamental solution in the normal to the boundary direction.

We examine now the integrals over the j-th element. Using the linear approximation of Eq. (5.6) for the boundary quantities u and u_n , the line integral appearing in the first sum of Eq. (5.8) may be written as



Figure 5.6 Modeling of the boundary with continuous linear elements.

$$\int_{\Gamma_{j}} v \, u_{n} \, ds = \int_{-1}^{1} v \left[\psi_{1}(\xi) \, u_{n}^{1} + \psi_{2}(\xi) \, u_{n}^{2} \right] \frac{\ell_{j}}{2} \, d\xi$$
$$= u_{n}^{1} \int_{-1}^{1} v \, \psi_{1}(\xi) \frac{\ell_{j}}{2} \, d\xi + u_{n}^{2} \int_{-1}^{1} v \, \psi_{2}(\xi) \frac{\ell_{j}}{2} \, d\xi$$
$$= g_{1}^{ij} \, u_{n}^{1} + g_{2}^{ij} \, u_{n}^{2}$$
(5.9)

where

$$g_1^{ij} = \frac{\ell_j}{2} \int_{-1}^{1} v \,\psi_1(\xi) \,d\xi \tag{5.10a}$$

$$g_2^{ij} = \frac{\ell_j}{2} \int_{-1}^{1} v \,\psi_2(\xi) \,d\xi \tag{5.10b}$$

and

$$v = \frac{1}{2\pi} \ell nr \tag{5.11}$$

$$r = \sqrt{\left[x(\xi) - x_i\right]^2 + \left[y(\xi) - y_i\right]^2}$$
(5.12)

It should be made clear that the superscript i in the symbols g_1^{ij} and g_2^{ij} , indicates the *i*-th node (p_i) where the source is applied, while the superscript j indicates the element over which the integration is carried out. Finally, the subscripts 1 and 2 denote in local numbering the points j and (j + 1), respectively.

In a similar fashion, the line integral appearing in the second sum of Eq. (5.8) may be written as

$$\int_{\Gamma_j} v_n u \, ds = h_1^{ij} u^1 + h_2^{ij} u^2 \tag{5.13}$$

where

$$h_{1}^{ij} = \frac{\ell_{j}}{2} \int_{-1}^{1} v_{n} \,\psi_{1}(\xi) \,d\xi \tag{5.14a}$$

$$h_2^{ij} = \frac{\ell_j}{2} \int_{-1}^{1} v_n \,\psi_2(\xi) \,d\xi \tag{5.14b}$$

and

$$v_n = \frac{\partial v}{\partial n} = \frac{1}{2\pi} \frac{\cos \phi}{r}$$
(5.15)

Substituting Eqs. (5.9) and (5.13) back into Eq. (5.8), the latter yields

$$-\varepsilon^{i}u^{i} + \sum_{j=1}^{N} \hat{H}_{ij} u^{j} = \sum_{j=1}^{N} G_{ij} u_{n}^{j}$$
(5.16)

where

$$\hat{H}_{ij} = \begin{cases} h_1^{i1} + h_2^{iN} & \text{for } j = 1\\ h_1^{ij} + h_2^{i,j-1} & \text{for } j = 2, 3, \dots, N \end{cases}$$
(5.17)

$$G_{ij} = \begin{cases} g_1^{i1} + g_2^{iN} & \text{for } j = 1\\ g_1^{ij} + g_2^{i,j-1} & \text{for } j = 2, 3, \dots, N \end{cases}$$
(5.18)

Equation (5.16) may be written in matrix form as

$$[H]\{u\} = [G]\{u_n\}$$
(5.19)

in which it has been set

$$[H] = -[\varepsilon] + [\hat{H}] \tag{5.20}$$

 $[\varepsilon]$ is a diagonal matrix with elements the coefficients ε' .

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Corner points and points of change in boundary conditions

In formulating Eq. (5.19), it was assumed that the quantities u and $u_n = \partial u/\partial n$ had a unique value. However, this is not always the case. For example, u_n is not continuous at the corner points, since its value is generally different before and after the corner. Likewise, in mixed boundary conditions different values are prescribed at nodal points, where the boundary conditions change type. At corner points, we may distinguish the following cases for the boundary conditions [7]:

- (a) Known: u_n before and after the corner Unknown: u at the corner
- (b) Known: u at the corner and un before the corner Unknown: un after the corner
- (c) Known: u at the corner and u_n after the corner Unknown: u_n before the corner
- (d) Known: *u* at the corner Unknown: *u_n before* and *after* the corner

In all the above cases, u is assumed to be continuous at the corner points, having therefore a unique value at these points. The terms *before* and *after* refer to the value of the quantity just before or right after the corner point according to the positive sense on the boundary (see Section 4.7).

The unknown boundary quantities are determined under the assumption that u_n may be discontinuous at all nodal points, which means that we are dealing with 2N values of u_n . With this in mind, we can write Eq. (5.19) as

$$[H]\{u\} = [G^*]\{u_u^*\}$$
(5.21)

where $\{u_n^*\}$ is a vector containing the 2N values of the normal derivative (two at each node) and $[G^*]$ is a $N \times 2N$ matrix whose elements are defined as

$$\left.\begin{array}{c}
G_{i,2j-1}^{\star} = g_{1}^{ij} \\
G_{i,2j}^{\star} = g_{2}^{ij}
\end{array}\right\} \quad (j = 1, 2, ..., N) \tag{5.22}$$

For the first three cases of the corner boundary conditions—cases (a), (b) and (c) only one boundary quantity is unknown. Therefore, rearranging the unknowns on the basis of the boundary conditions, Eq. (5.21) produces a system of N linear equations, which can be solved for the N unknown boundary quantities. The unknowns are rearranged by examining all the nodes. If the value of u is unknown at a node, then the respective column remains at the left-hand side of Eq. (5.21), otherwise this column is multiplied by the known value of u, its sign is switched and is shifted to the right-hand side of the equation. Similarly, if the value of u_n is unknown, then the respective column of $[G^*]$ is shifted with opposite sign to the lefthand side of the equation. The two consecutive columns corresponding to u_n are added together, if the node is not a corner point of the *actual* boundary. After completing this process, the right-hand side of the equation contains only known quantities and, thus, the matrix multiplication results in a single vector. Corner points may also be treated using one-sided discontinuous elements before and after the corner (Fig. 5.7). Thus, two separate nodes appear in the equations at which two different values of u_n are computed.

It should be noted that an abrupt change in the boundary's slope (especially reentrant corners) or a change in the type of boundary conditions causes a local singularity in the behavior of the solution, which may even "*pollute*" the numerical results over the whole domain. A remedy to this problem is the refinement of the elements near the point of singularity. Nevertheless, this technique is not always successful in giving a reliable solution, especially for its derivative, and recourse to special techniques is unavoidable [8].



Figure 5.7 Discontinuous elements adjacent to a corner ($u_n = \partial u / \partial n$).

5.4 Evaluation of line integrals on linear elements

The matrices [G] and [H] appearing in Eq. (5.19) require the computation of the line integrals (5.10) and (5.14) whose integrands are products of the fundamental solution v or its normal derivative $v_n = \frac{\partial v}{\partial n}$ and the linear shape functions $\psi_1(\xi)$ and $\psi_2(\xi)$. The integrations are carried out over the interval [-1,+1]. Two cases are considered for the linear elements as it was done for the constant elements. These cases are dictated by the behavior of the functions (5.11) and (5.15). Specifically, when the j-th element, over which the integration is performed, does not contain the source point i, i.e. $i \neq j$, then it is always $r \neq 0$ and the integral is regular. On the other hand, when the source point lies on the j-th element, i.e. i = j, then the distance r takes also the value r = 0 and the behavior of the integral is singular. The integration for the first case $(i \neq j)$ will be referred to as *outside integration*, whereas for the second case (i = j) as *inside integration*.

5.4.1 Outside integration

The integrals (5.10) and (5.14) may be evaluated analytically using symbolic languages (e.g. MAPLE). This process, however, yields very lengthy expressions, which in some cases may cover several pages, making them computationally impractical. A very practical and accurate approach is the numerical integration. Any integration rule may be utilized for this purpose, e.g. trapezoidal rule, Simpson's rule or Newton-Cotes integration formulae. But, the most suitable method for the numerical evaluation of BEM integrals is the Gaussian quadrature (see Appendix B). This method approximates the integral with great accuracy using the least number of values of the integrand. The numerical integration should not be performed "blindly". The accuracy depends not only on the number of integration points, but also on how the integrand varies within the integration interval. A smooth variation of the integrand gives more accurate results. Therefore, the integration process requires a thoughtful consideration and special care is required when the integrand exhibits intense changes. In integrals (5.10) and (5.14), the shape functions vary smoothly and consequently, the behavior of the integrand is dominated by the functions ℓnr and 1/r.



Figure 5.8 Points A, B and C where the source is applied.

In order to have a better insight into the variation of the integrand, we consider the domain of Fig. 5.8 and examine the function $g(\xi) = \psi_1(\xi) \ell nr$ on the element with extreme points (4,4) and (3,4). Three locations for the source are considered:

- (i) location A relatively far from the element,
- (ii) location B at a relatively moderate distance from the element, and

(iii) location C relatively close to the element.

For the particular element, the transform equations (5.4a) and (5.4b) become

$$x(\xi) = \left[\frac{1}{2}(1-\xi)\right]4 + \left[\frac{1}{2}(1+\xi)\right]3 = 3.5 - 0.5\xi$$

$$y(\xi) = \left[\frac{1}{2}(1-\xi)\right]4 + \left[\frac{1}{2}(1+\xi)\right]4 = 4$$

and according to Eq. (5.12), the relative distance from the source point is

$$r(\xi) = \sqrt{[x(\xi) - x_i]^2 + [y(\xi) - y_i]^2}$$

where x_i and y_i are the global coordinates of the source point.



Figure 5.9 Behavior of the integrand $g(\xi) = \psi_1(\xi) \ell nr$ for different locations of the source.

The behavior of the integrand associated with points A, B and C is illustrated graphically through the plots of $g(\xi)$ depicted in Figure 5.9. It should be noticed that the variation of the function $g(\xi)$ is quite different when the source is located close to the element over which the integration is performed. Therefore, an efficient programming of BEM should take into account such a behavior by using Gaussian quadratures with variable number of integration points (i.e., increasing number of points as the source is getting closer to the element). The number of integration points should be chosen in a way to ensure *sufficient accuracy*. An unnecessarily, however, large number of integration points should be avoided, in order to keep the computation cost low.

5.4.2 Inside integration

In this case the source lies on the element over which the integration is performed. As the integration point runs along the whole element, it will coincide inevitably with the source point. There, the distance r vanishes and the integrands of Eqs. (5.10) and (5.14) exhibit a singular behavior, because the factors lnr and $\cos \phi/r$ become infinite for r = 0 (see Eqs. (5.11) and (5.15)). These integrals are known as *singular integrals*. Their value exists and is determined through special integration techniques, either analytical or numerical. Even an *indirect* method has been invented to circumvent the evaluation of the singular integrals by computing directly the singular coefficients H_{ii} and G_{ii} (see Section 5.4.3). In the sequel, we will first study the integrals with *logarithmic singularity*, and then those with *Cauchy type singularity* (1/r).

5.4.2.1 Integrals with logarithmic singularity

(a) Analytical integration

We consider the general case of the discontinuous linear element shown in Fig. 5.10. The continuous element results as a special case, when the nodes are shifted to the extreme points. The two nodes of the linear element are assigned locally the numbers 1 and 2, and their global coordinates are denoted by (x_1, y_1) and (x_2, y_2) . Using this notation, it can be easily proved that the coordinate transformation from the local to the global system is expressed by the equations

$$x(\xi) = \frac{\kappa_1 x_2 + \kappa_2 x_1}{\kappa} + \frac{x_2 - x_1}{\kappa} \xi$$
(5.23a)

$$y(\xi) = \frac{\kappa_1 y_2 + \kappa_2 y_1}{\kappa} + \frac{y_2 - y_1}{\kappa} \xi$$
(5.23b)

where $-1 \le \xi \le 1$ and

$$\kappa = \kappa_1 + \kappa_2, \qquad \xi = \frac{x'}{\ell/2}$$
(5.24)

Furthermore, Eqs. (5.23) may be rewritten as

$$x(\xi) = \psi_1(\xi) x_1 + \psi_2(\xi) x_2$$

$$y(\xi) = \psi_1(\xi) y_1 + \psi_2(\xi) y_2$$

in which the shape functions $\psi_1(\xi)$ and $\psi_2(\xi)$ are given by the expressions

$$\psi_1(\xi) = \frac{1}{\kappa} (\kappa_2 - \xi) \tag{5.25a}$$

$$\psi_2(\xi) = \frac{1}{\kappa}(\kappa_1 + \xi) \tag{5.25b}$$

with $0 \leq \kappa_1, \kappa_2 \leq 1$.



Figure 5.10 Discontinuous linear element in local coordinate systems.

It is apparent that for $\kappa_1 = \kappa_2 = 1$ the shape functions of Eqs. (5.25) reduce to Eqs. (5.7), which represent the shape functions of the continuous linear element.

If the source lies on the J-th (J = 1, 2) local node of the element, its coordinates are going to be

$$x_{J} = \frac{\kappa_{1} x_{2} + \kappa_{2} x_{1}}{\kappa} + \frac{x_{2} - x_{1}}{\kappa} \xi_{J}$$
$$y_{J} = \frac{\kappa_{1} y_{2} + \kappa_{2} y_{1}}{\kappa} + \frac{y_{2} - y_{1}}{\kappa} \xi_{J}$$

where

 $\xi_1 = -\kappa_1 \,, \qquad \xi_2 = \kappa_2$

and the relative distance of Eq. (5.12) becomes

$$r(\xi) = \frac{\ell_j}{2} |\xi - \xi_J|$$
(5.26)

where ℓ_j is the length of the *j*-th element.

Having described the discontinuous linear element, we will study now the integrals of Eqs. (5.10) for the case where the source node *i* lies on the integration element *j*. The integral (5.10a) may then be written for a discontinuous element as

$$\frac{4\pi}{\ell_j} g_1^{ij} = \int_{-1}^{1} \psi_1(\xi) \,\ell nr \,d\xi$$

= $\int_{-1}^{\xi_j} \psi_1(\xi) \,\ell nr \,d\xi + \int_{\xi_j}^{1} \psi_1(\xi) \,\ell nr \,d\xi$
= $I_1 + I_2$ (5.27)

where, the *i*-th node coincides with the J-th local node (J = 1, 2) of element j (j = 1, 2, ..., N). Note, that for discontinuous elements the number of nodes cannot be equal to the number of elements N. For example, if all the elements are discontinuous, the total number of nodes will be 2N, which is the maximum possible number of nodes for a linear element discretization.

The transformation

$$\xi = -(1 + \xi_J)z + \xi_J \tag{5.28}$$

maps the integration interval $[-1, \xi_J]$ of I_1 onto the interval [0, +1]. Substituting into the expression of I_1 which is given in Eq. (5.27), we have

$$I_{1} = \int_{-1}^{\xi_{J}} \psi_{1}(\xi) \,\ell nr \,d\xi$$

= $\int_{0}^{1} \frac{1}{\kappa} \Big[(\kappa_{2} - \xi_{J}) + (1 + \xi_{J}) \,z \Big] \,\ell n \Big[\frac{\ell_{J}}{2} (1 + \xi_{J}) \,z \Big] (1 + \xi_{J}) \,dz$

Introducing the quantities

$$\theta = \frac{\ell_J}{2} (1 + \xi_J) z, \quad \theta_1 = \frac{\ell_J}{2} (1 + \xi_J)$$

the above integral becomes

$$I_1 = \int_0^{\theta_1} \frac{1}{\kappa} \frac{2}{\ell_j} \left[(\kappa_2 - \xi_J) + \frac{2}{\ell_j} \theta \right] \ell n \theta \, d\theta \tag{5.29}$$

which is of the form

$$I_1 = \int_0^{\theta_1} (a+b\theta) \,\ell n\theta \,d\theta \tag{5.30}$$

where the two constants are

$$a = \frac{1}{\kappa} \frac{2}{\ell_j} (\kappa_2 - \xi_J)$$
 and $b = \frac{1}{\kappa} \left(\frac{2}{\ell_j}\right)^2$

The integral of Eq. (5.30) can be readily integrated by parts to yield

$$I_1 = \left[a \,\theta \left(\ell n \theta - 1 \right) + b \,\theta^2 \left(\frac{1}{2} \,\ell n \theta - \frac{1}{4} \right) \right]_0^{\theta_1} \tag{5.31}$$

or using the definitions of the constants a and b, it results in

$$I_{1} = \frac{1}{\kappa} (\kappa_{2} - \xi_{J})(1 + \xi_{J}) \left\{ \ell n \left[\frac{\ell_{J}}{2} (1 + \xi_{J}) \right] - 1 \right\} + \frac{1}{\kappa} (1 + \xi_{J})^{2} \left\{ \frac{1}{2} \ell n \left[\frac{\ell_{J}}{2} (1 + \xi_{J}) \right] - \frac{1}{4} \right\}$$
(5.32)

Referring now to the second integral of Eq. (5.27), the transformation

$$\xi = (1 - \xi_J) z + \xi_J \tag{5.33}$$

maps the integration interval $[\xi_J, +1]$ of I_2 onto the interval [0, +1]. Thus, the new expression of the integral is

$$I_{2} = \int_{\xi_{J}}^{1} \psi_{1}(\xi) \, \ell nr \, d\xi$$

= $\int_{0}^{1} \frac{1}{\kappa} \left[(\kappa_{2} - \xi_{J}) - (1 - \xi_{J}) \, z \right] \, \ell n \left[\frac{\ell_{J}}{2} (1 - \xi_{J}) \, z \right] (1 - \xi_{J}) \, dz$

Furthermore, by setting

$$\theta = \frac{\ell_J}{2} (1 - \xi_J) z, \qquad \theta_1 = \frac{\ell_J}{2} (1 - \xi_J)$$

the above integral may also be written as

$$I_2 = \int_0^{\theta_1} \frac{1}{\kappa} \frac{2}{\ell_j} \left[(\kappa_2 - \xi_J) - \frac{2}{\ell_j} \theta \right] \ell n \theta \, d\theta \tag{5.34}$$

The integral of Eq. (5.34) has exactly the same form as that of Eq. (5.30), but with different values of the two constants, which are now defined as

$$a = rac{1}{\kappa} rac{2}{\ell_j} \left(\kappa_2 - \xi_J
ight) ext{ and } b = -rac{1}{\kappa} \left(rac{2}{\ell_j}
ight)^2$$

Substituting this set of constants into Eq. (5.31), we obtain the final expression for the second integral of Eq. (5.27)

$$I_{2} = \frac{1}{\kappa} (\kappa_{2} - \xi_{J}) (1 - \xi_{J}) \left\{ \ell n \left[\frac{\ell_{J}}{2} (1 - \xi_{J}) \right] - 1 \right\} - \frac{1}{\kappa} (1 - \xi_{J})^{2} \left\{ \frac{1}{2} \ell n \left[\frac{\ell_{J}}{2} (1 - \xi_{J}) \right] - \frac{1}{4} \right\}$$
(5.35)

The results of Eqs. (5.32) and (5.35) are combined into Eq. (5.27) to produce the analytical expression for the influence coefficient g_1^{ij} when the node *i* lies on the element *j* (i.e., singular case).

The other influence coefficient defined by Eq. (5.10b), is written for a discontinuous element as

$$\frac{4\pi}{\ell_j} g_2^{ij} = \int_{-1}^{1} \psi_2(\xi) \,\ell nr \,d\xi$$
$$= \int_{-1}^{\xi_J} \psi_2(\xi) \,\ell nr \,d\xi + \int_{\xi_J}^{1} \psi_2(\xi) \,\ell nr \,d\xi$$
$$= I_3 + I_4$$
(5.36)

where the indices i, j, and J are defined as for the case of Eq. (5.27). The integrals I_3 and I_4 are evaluated following the procedure applied for I_1 and I_2 . The resulting expressions for these two integrals are

$$I_{3} = \frac{1}{\kappa} (\kappa_{1} + \xi_{J}) (1 + \xi_{J}) \left\{ \ell n \left[\frac{\ell_{J}}{2} (1 + \xi_{J}) \right] - 1 \right\} - \frac{1}{\kappa} (1 + \xi_{J})^{2} \left\{ \frac{1}{2} \ell n \left[\frac{\ell_{J}}{2} (1 + \xi_{J}) \right] - \frac{1}{4} \right\}$$
(5.37)

and

$$I_{4} = \frac{1}{\kappa} (\kappa_{1} + \xi_{J}) (1 - \xi_{J}) \left\{ \ell n \left[\frac{\ell_{J}}{2} (1 - \xi_{J}) \right] - 1 \right\} + \frac{1}{\kappa} (1 - \xi_{J})^{2} \left\{ \frac{1}{2} \ell n \left[\frac{\ell_{J}}{2} (1 - \xi_{J}) \right] - \frac{1}{4} \right\}$$
(5.38)

which may be combined into Eq. (5.36) to give the analytical expression for the singular case of the influence coefficient g_2^{ij} .

The influence coefficients for continuous elements may be deduced from the foregoing equations by setting $\xi_1 = -1$ and $\xi_2 = 1$. In this case, Eqs. (5.27) and (5.36) give: (i) for $\xi_1 = -1$, it is $\kappa_1 = \kappa_2 = 1$, $\kappa = 2$ and

$$g_{1}^{ij} = \frac{\ell_{j}}{4\pi} (\ell n \ell_{j} - 1.5)$$

$$g_{2}^{ij} = \frac{\ell_{j}}{4\pi} (\ell n \ell_{j} - 0.5)$$
(5.39a)

(ii) for $\xi_J = +1$, it is $\kappa_1 = \kappa_2 = 1$, $\kappa = 2$ and

$$g_{1}^{ij} = \frac{\ell_{j}}{4\pi} (\ell n \ell_{j} - 0.5)$$

$$g_{2}^{ij} = \frac{\ell_{j}}{4\pi} (\ell n \ell_{j} - 1.5)$$
(5.39b)

(b) Numerical integration

According to the previous discussion, integrals with logarithmic singularity, like those of Eqs. (5.27) and (5.36), can be set in the form

$$I = \int_{0}^{1} f(x) \, \ell nx \, dx \tag{5.40}$$

Special Gauss integration schemes have been developed for its numerical evaluation. Stroud and Secrest [10] approximated this type of integral as follows

$$\int_{0}^{1} f(\xi) \, \ell n\left(\frac{1}{\xi}\right) d\xi \approx \sum_{k=1}^{n} f(\xi_k) \, w_k \tag{5.41}$$

and produced tables with the integration points ξ_k and the corresponding weights w_k (see Appendix B and Refs. [10,11]). It should be emphasized here that the integration interval must always be reduced to [0,+1]. Therefore, when the source lies inside the element, the integration interval must be split into two intervals, $[-1,\xi_J]$ and $[\xi_J,+1]$. Consequently, the influence coefficients of Eqs. (5.10) become

$$\frac{4\pi}{\ell_j} g_{\alpha}^{ij} = \int_{-1}^1 \psi_{\alpha}(\xi) \,\ell nr \,d\xi$$
$$= \int_{-1}^{\xi_j} \psi_{\alpha}(\xi) \,\ell nr \,d\xi + \int_{\xi_j}^1 \psi_{\alpha}(\xi) \,\ell nr \,d\xi \qquad (\alpha = 1, 2)$$
(5.42)

where ξ_j (J = 1, 2) denotes the node of the *j*-th element where the source node *i* is located and *r* is the relative distance between the source and the integration point given in Eq. (5.26).

The transformations (5.28) and (5.33) map the two integration intervals $[-1, \xi_J]$ and $[\xi_J, +1]$ of Eq. (5.42) onto the interval [0, +1], respectively. Equation (5.42) is then written as

$$\begin{aligned} \frac{4\pi}{\ell_j} g_{\alpha}^{ij} &= \int_0^1 \psi_{\alpha}^{(1)}(z) \left(1 + \xi_J\right) \ell n \left[\frac{\ell_j}{2} (1 + \xi_J) z\right] dz \\ &+ \int_0^1 \psi_{\alpha}^{(2)}(z) \left(1 - \xi_J\right) \ell n \left[\frac{\ell_j}{2} (1 - \xi_J) z\right] dz \end{aligned}$$

or by expanding the logarithms

$$\frac{4\pi}{\ell_{j}}g_{\alpha}^{ij} = (1+\xi_{J}) \ell n \left[\frac{\ell_{j}}{2}(1+\xi_{J})\right] \int_{0}^{1} \psi_{\alpha}^{(1)}(z) dz
+ (1+\xi_{J}) \int_{0}^{1} \psi_{\alpha}^{(1)}(z) \ell nz dz
+ (1-\xi_{J}) \ell n \left[\frac{\ell_{j}}{2}(1-\xi_{J})\right] \int_{0}^{1} \psi_{\alpha}^{(2)}(z) dz
+ (1-\xi_{J}) \int_{0}^{1} \psi_{\alpha}^{(2)}(z) \ell nz dz
= I_{1} + I_{2} + I_{3} + I_{4}$$
(5.43)

where $\psi_{\alpha}^{(1)}(z)$ and $\psi_{\alpha}^{(2)}(z)$ ($\alpha = 1, 2$) are transformed shape functions obtained from $\psi_{\alpha}(\xi)$ by expressing the variable ξ in terms of z according to Eq. (5.28) and Eq. (5.33), respectively. The integrals I_1 and I_3 are regular and can be evaluated either analytically or by applying the conventional Gauss integration, while the I_2 and I_4 are singular and can be evaluated numerically by virtue of Eq. (5.41).

(c) Integration by extracting the singularity

The integrals of Eqs. (5.10) may also be written as

$$\frac{4\pi}{\ell_{j}}g_{\alpha}^{ij} = \int_{-1}^{1}\psi_{\alpha}(\xi)\,\ell nr\,d\xi$$
$$= \int_{-1}^{1}[\psi_{\alpha}(\xi) - \psi_{\alpha}(\xi_{J})]\,\ell nr\,d\xi + \psi_{\alpha}(\xi_{J})\int_{-1}^{1}\ell nr\,d\xi$$
(5.44)

The integrand of the first integral vanishes for $\xi = \xi_J$. Indeed, this integrand can be written as

$$rac{\ell nr(\xi)}{\left[rac{1}{\psi_lpha(\xi)-\psi_lpha(\xi_J)}
ight]}$$

It can be easily noticed that this expression takes for $\xi = \xi_J$ the indeterminate form \Re . Thus, applying consecutively L'Hôspital's rule and taking into account that the derivatives $\psi'_{\alpha}(\xi) = d\psi_{\alpha}(\xi)/d\xi$ and $r'(\xi) = dr(\xi)/d\xi$ are finite and do not vanish, the integrand in question yields

$$\lim_{\xi \to \xi_J} \left\{ \left[\psi_{\alpha}(\xi) - \psi_{\alpha}(\xi_J) \right] \ell nr(\xi) \right\} = 0$$

This result shows that the first integral of Eq. (5.44) is regular and can be evaluated using conventional Gaussian quadrature. It is worth mentioning that the accuracy of this numerical integration is increased if it is carried out on two separate subintervals, $[-1, \xi_J]$ and $[\xi_J, +1]$. Apparently, this is dictated by the shape of the integrand as it is depicted in Fig. 5.11. The second integral of Eq. (5.44) exhibits a logarithmic singularity, but it can be readily evaluated analytically. In closing we could say that the method of *extracting the singularity* simplifies the evaluation of the integral (5.44), though it does not avoid the evaluation of singular integrals.



Figure 5.11 Variation of the function $f(\xi) = [\psi_1(\xi) - \psi_1(\xi_J)] lnr(\xi)$ for $\xi_J = -0.5$.

Table 5.1	Values of the integrals	g^y_{α} , when	the source	<i>i</i> lies	on the
	element.				

Influence Coefficient	Analytical integration	Gauss integration, Eq. (5.43)	Extracting the singularity, Eq. (5.44)
g_1^{1j}	-0.2970889377	-0.2970889376	-0.2970889377
g_2^{1j}	0.0274675251	0.0274675250	0.0274675251
g_1^{2j}	0.0274675251	0.0274675250	0.0274675251
g_2^{2j}	-0.2970889377	-0.2970889376	-0.2970889377

Table 5.1 presents the values of the integrals g_{α}^{ij} . They have been computed: (a) analytically from Eqs. (5.27) and (5.36), (b) using the special Gauss integration of Eq. (5.41) with 8 integration points, and (c) by extracting the singularity according to Eq. (5.44). In the last case, the regular integral was computed employing eight Gauss points on each of the subintervals $[-1, \xi_J]$ and $[\xi_J, +1]$, while the singular one was evaluated analytically. For all cases the element data are $x_1 = 3.0$, $y_1 = 2.0$, $x_2 = 1.0$, $y_2 = 3.0$, and $\kappa_1 = \kappa_2 = 0.5$.

5.4.2.2 Integrals with Cauchy-type singularity

It has already been mentioned, that the computation of the diagonal elements \hat{H}_{ii} and G_{ii} of matrices [H] and [G] requires an inside integration. In the preceding subsection we presented techniques for the evaluation of the coefficients G_{ii} which are line integrals with logarithmic singularity. The coefficients \hat{H}_{ii} , however, are determined by evaluating integrals of the form (see Eqs. (5.14) and (5.15))

$$\int_{-1}^{1} \psi_{\alpha}(\xi) \, \frac{\cos \phi}{r} \, d\xi \,, \qquad r = \frac{\ell_{j}}{2} \, |\xi - \xi_{j}| \tag{5.45}$$

where $\psi_{\alpha}(\xi)$ ($\alpha = 1, 2$) are given in Eqs. (5.25) and r in Eq. (5.26).

Linear elements approximate the geometry by a straight line, so apparently it is

$$\cos\phi = \cos(\pm\frac{\pi}{2}) = 0$$

along the whole element, since $\phi = angle(\mathbf{r}, \mathbf{n})$ (see Appendix A). Therefore, the integrand of Eq. (5.45) becomes

$$\psi_{\alpha}(\xi) \frac{\cos \phi}{r} = \begin{cases} 0 , & \xi \neq \xi_J \\ \frac{0}{0} , & \xi = \xi_J \end{cases}$$

Applying L'Hôspital's rule, the above expression yields

$$\lim_{\xi \to \xi_J} \frac{\psi_{\alpha}(\xi) \cos \phi}{r(\xi)} = \lim_{\xi \to \xi_J} \frac{\psi_{\alpha}'(\xi) \cos \phi}{r'(\xi)} = 0$$
(5.46)

because the derivative $\psi'_{\alpha}(\xi) = (-1)^{\alpha}/\kappa$ is constant (see Eq. 5.25) and finite in the interval [-1,+1], and the same is true for the derivative $r'(\xi)$, which according to Eq. (5.26) is given as

$$r'(\xi) = \frac{\ell_j}{2} \operatorname{sign}(\xi)$$
(5.47)

where sign(ξ) (signum of ξ) is the function defined as

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sign(
$$\xi$$
) =

$$\begin{cases}
+1, & \xi > 0 \\
-1, & \xi < 0
\end{cases}$$
(5.48)

Consequently, the value of integral (5.45) is

$$\int_{-1}^{1} \psi_{\alpha}(\xi) \, \frac{\cos \phi}{r} \, d\xi = 0 \tag{5.49}$$

It should be noted that this result is not valid for higher order elements, since for these elements $\cos \phi(\xi) \neq 0$.

The integrand in (5.45) behaves as 1/r and becomes infinite when r = 0. This singularity is known as *Cauchy type singularity*. Analytical, numerical as well as hybrid techniques have been developed for the evaluation of integrals with this type of singularity. Their use and programming require special care. In addition to the diagonal elements \hat{H}_{ii} , the coefficients ε^i must also be computed (see Eq. 5.16 or 5.20), which, of course, increases the computational task. However, it is possible to evaluate directly the elements $H_{ii} = \hat{H}_{ii} - \varepsilon_i$ by an indirect method that avoids the evaluation of any singular integral.

5.4.3 Indirect evaluation of the diagonal influence coefficients

The matrices [G] and [H] in Eq. (4.7) for constant elements, or in Eq. (5.19) for linear elements, are affected only by the boundary geometry, its discretization and the employed type of element. Hence, these matrices do not depend on the boundary conditions, that is, they remain unchanged for given boundary geometry, boundary discretization and type of boundary elements.

The indirect integration is based on the fact that the function $u = \alpha x + by + c$ is a solution to the Laplace equation, namely

 $\nabla^2 u = 0 \quad \text{in } \ \Omega$

with boundary conditions

 $u = \alpha x + b y + c \quad \text{on} \quad \Gamma \tag{5.50a}$

and

$$u_n = \nabla u \cdot \mathbf{n} = \alpha \, n_x + b \, n_y \quad \text{on} \quad \Gamma \tag{5.50b}$$

(a) Evaluation of the elements H_{ii}

Let us assume that a = b = 0 and c = 1. In this case, it will be u = 1 and $u_n = 0$ on the boundary. Obviously, these values must satisfy Eq. (5.19), which gives after substitution

 $[H]{1} = 0$

where $\{1\}$ is the vector whose elements are all equal to 1. The above equation may also be written as

$$\sum_{j=1}^{N} H_{ij} = 0$$

or

$$H_{ii} = -\sum_{\substack{j=1\\i\neq j}}^{N} H_{ij} \qquad (i = 1, 2, ..., N)$$
(5.51)

Equation (5.51) states that the diagonal element in the *i*-th row of matrix [H] is equal to the negative sum of the remaining elements in this row. It should be noted that Eq. (5.51) is valid only for a closed domain Ω . For infinite domains, it is not valid, because a constant value for u violates the regularity condition at the infinity. Nevertheless, it is possible even in this case to evaluate H_{ii} using an indirect approach (see Ref. [12]).

(b) Evaluation of the elements G_n

Once the matrix [H] has been computed, the diagonal elements G_{ii} of matrix [G] can also be computed using an indirect method, which avoids the evaluation of any integrals, either regular or singular. To this end, Eq. (5.19) is applied for the function

u = a x + b y

giving,

$$[G]\{u_n\} = [H]\{u\}$$

or

$$\sum_{j=1}^{N} G_{ij} u_{n}^{j} = \sum_{j=1}^{N} H_{ij} u^{j} \qquad (i = 1, 2, \dots, N)$$

The above equation is solved for G_{ii} , yielding

$$G_{ii} = \frac{1}{u_n^i} \left(-\sum_{\substack{j=1\\j\neq i}}^N G_{ij} \, u_n^j + \sum_{j=1}^N H_{ij} \, u^j \right)$$
(5.52)

The boundary values u^{j} and u_{n}^{j} appearing in Eq. (5.52) are computed from the following expressions

$$u^{j} = a x_{j} + b y_{j} \tag{5.53a}$$

$$u_n^j = a \, n_x^j + b \, n_y^j \tag{5.53b}$$

where (x_j, y_j) are the coordinates of the j-th node, and (n_x^j, n_y^j) are the components of the unit vector normal to the j-th element. The constants a and b are chosen arbitrarily, but under the condition that $u_n^i \neq 0$. This means according to Eq. (5.53b), that the vector with components the coefficients a and b must not be normal to the vector \mathbf{n}^j (j = 1, 2, ..., N), or, in other words, must not be parallel to the j-th boundary element. This may be achieved by setting a = 1, $b = \lambda > 0$ and choosing λ so that

$$\lambda \neq \left| \frac{y_{j+1} - y_j}{x_{j+1} - x_j} \right| \tag{5.54}$$

5.5 Higher order elements

Constant and linear elements can not approximate with sufficient accuracy the geometry of curvilinear boundaries. For this reason it is recommended to use curvilinear elements, for which the interpolating polynomials are of degree higher than one. In general, their form in the normalized interval [-1, +1] is going to be

$$f(\xi) = a_o + a_1\xi + a_2\xi^2 + a_3\xi^3 + \dots + a_n\xi^n \quad (-1 \le \xi \le 1)$$
(5.55)

For n = 2, Eq. (5.55) yields the interpolation function for the *quadratic* or *parabolic* element, for n = 3 that of the *cubic* element and so forth. In what it follows we will limit our presentation to the parabolic element. For higher order elements or for a general theory on isoparametric elements the reader is advised to look in Ref. [13].

The boundary quantities u and u_n in the line integrals of Eq. (5.8) are functions of the arc length measured from some origin. When a parabolic variation is assumed the quantities u or u_n will be expressed by a polynomial of the form

$$f(s) = \alpha_o + \alpha_1 s + \alpha_2 s^2 \tag{5.56}$$

The coordinates of point $(x, y) \in \Gamma$, which varies during the integration, are also functions of s, i.e., x = x(s), y = y(s). Hence

$$r = \sqrt{[x(s) - x_i]^2 + [y(s) - y_i]^2} = r(s)$$

and

$$v=v(s)\,, \qquad v_n=rac{\partial v}{\partial n}=v_n(s)$$
Consequently, the integral to be evaluated is of the form

$$I = \int_{\Gamma_j} w(s) ds \tag{5.57}$$

Of course, the integration can be performed by first establishing the transformation $s = s(\xi)$ and then substituting it in the integral (5.57). Although this method is conceptually simple, its implementation requires the establishment of complicated expressions of s and for this reasons it is not the most suitable. Instead, another method is presented below, which simplifies considerably the integration procedure.

The integration is performed over the normalized interval [-1,+1] with respect to the intrinsic coordinate ξ , and thus, the integral (5.57) becomes

$$I = \int_{-1}^{1} w^{*}(\xi) |J(\xi)| d\xi$$
(5.58)

where $|J(\xi)|$ is the Jacobian of the transformation which maps the parabolic arc Γ_j of the xy-plane onto the straight line segment with $-1 \le \xi \le 1$ and $\eta = 0$ of the $\xi\eta$ -plane (see Fig. 5.12).



Figure 5.12 Parabolic element in global and local coordinate systems.

The boundary quantity f (u or u_n) is approximated directly in the interval [-1,+1] by a second order polynomial in ξ , namely,

$$f(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 \tag{5.59}$$

The coefficients α_0 , α_1 and α_2 are determined from the requirement that the function $f(\xi)$ takes the nodal values f_1 , f_2 , f_3 at points $\xi = -1, 0, 1$, respectively (see Fig 5.12). Hence,

$$\begin{cases}
f(-1) = f_1 \\
f(0) = f_2 \\
f(1) = f_3
\end{cases}$$
(5.60)

Applying conditions (5.60) to Eq. (5.59), we get the following system of equations for the unknown coefficients

$$\alpha_0 - \alpha_1 + \alpha_2 = f_1$$

$$\alpha_0 = f_2$$

$$\alpha_0 + \alpha_1 + \alpha_2 = f_3$$

whose solution is

$$\begin{array}{c} \alpha_{0} = f_{2} \\ \alpha_{1} = \frac{f_{3} - f_{1}}{2} \\ \alpha_{2} = \frac{f_{1} - 2f_{2} + f_{3}}{2} \end{array} \right\}$$

$$(5.61)$$

Introducing Eqs. (5.61) into Eq. (5.59), we obtain the expression of the boundary quantity in terms of the three element nodal values

$$f(\xi) = f_2 + \frac{f_3 - f_1}{2} \xi + \frac{f_1 - 2f_2 + f_3}{2} \xi^2$$
(5.62)

Equation (5.62) may further be written in the form

$$f(\xi) = \psi_1(\xi) f_1 + \psi_2(\xi) f_2 + \psi_3(\xi) f_3$$
(5.63)

or

$$f(\xi) = \sum_{\alpha=1}^{3} \psi_{\alpha}(\xi) f_{\alpha}$$
(5.64)

where

$$\psi_{1}(\xi) = -\frac{1}{2}\xi(1-\xi)$$

$$\psi_{2}(\xi) = (1-\xi)(1+\xi)$$

$$\psi_{3}(\xi) = \frac{1}{2}\xi(1+\xi)$$
(5.65)

The functions defined in Eqs. (5.65) are the *shape functions* of the parabolic or quadratic element.

The mapping of the parabolic element from the xy-plane onto the interval $-1 \le \xi \le 1$ of the $\xi\eta$ -plane is accomplished through the transformation

$$x(\xi) = b_0 + b_1 \xi + b_2 \xi^2 y(\xi) = c_0 + c_1 \xi + c_2 \xi^2$$
(5.66)

We can readily conclude from Eqs. (5.59) and (5.66) that the parabolic element at hand is isoparametric, since both the geometry and the boundary quantity are approximated by polynomials of the same degree. The coefficients b_k and c_k (k = 0, 1, 2) in Eqs. (5.66) are evaluated from the requirement that the element arc should pass through the points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) for $\xi = -1, 0, 1$, respectively. These conditions are expressed mathematically as

$$x(-1) = x_1$$
, $x(0) = x_2$, $x(1) = x_3$
 $y(-1) = y_1$, $y(0) = y_2$, $y(1) = y_3$

It is apparent that the above conditions yield expressions for $x(\xi)$ and $y(\xi)$, which are similar to those of Eq. (5.64). More specifically,

$$x(\xi) = \sum_{\alpha=1}^{3} \psi_{\alpha}(\xi) x_{\alpha}$$

$$y(\xi) = \sum_{\alpha=1}^{3} \psi_{\alpha}(\xi) y_{\alpha}$$
(5.67)

where the shape functions $\psi_{\alpha}(\xi)$ are given in Eq. (5.65).

Equations (5.67) may be employed to express the distance r, as well as the kernels v(r) and $v_n(r)$ of the integrals representing the influence coefficients, as functions of the variable ξ . Finally, the Jacobian of Eq. (5.58) is evaluated from the expression

$$ds = \sqrt{dx^{2} + dy^{2}} = \sqrt{[x'(\xi)]^{2} + [y'(\xi)]^{2}} d\xi$$

Hence

$$|J(\xi)| = \left\{ \left[x'(\xi) \right]^2 + \left[y'(\xi) \right]^2 \right\}^{1/2}$$

= $\left[(b_{\mathfrak{k}} + 2b_2\xi)^2 + (c_1 + 2c_2\xi)^2 \right]^{1/2}$ (5.68)

where on the basis of Eq. (5.61), it is

$$b_{1} = \frac{x_{3} - x_{1}}{2}, \qquad b_{2} = \frac{x_{1} - 2x_{2} + x_{3}}{2}$$

$$c_{1} = \frac{y_{3} - y_{1}}{2}, \qquad c_{2} = \frac{y_{1} - 2y_{2} + y_{3}}{2}$$
(5.69)

~

After discretizing the boundary into N parabolic elements, the boundary integral equation (3.31) may be written as

$$\varepsilon^{i}u^{i} = -\sum_{j=1}^{N} \int_{\Gamma_{j}} vu_{n} \, ds + \sum_{j=1}^{N} \int_{\Gamma_{j}} uv_{n} \, ds \tag{5.70}$$

Taking into account the parabolic variation of the boundary quantities on the elements, the second line integral in the right-hand side of Eq. (5.70) yields

$$\int_{\Gamma_{j}} u(q) v_{n}(p_{i},q) ds_{q} = \int_{\Gamma_{j}} (\psi_{1}u^{1} + \psi_{2}u^{2} + \psi_{3}u^{3}) v_{n} ds$$
$$= h_{1}^{ij}u^{1} + h_{2}^{ij}u^{2} + h_{3}^{ij}u^{3}$$
(5.71)

where it has been set

$$\begin{array}{c} h_{1}^{ij} = \int_{\Gamma_{j}} \psi_{1} v_{n} \, ds \\ h_{2}^{ij} = \int_{\Gamma_{j}} \psi_{2} v_{n} \, ds \\ h_{3}^{ij} = \int_{\Gamma_{j}} \psi_{3} v_{n} \, ds \end{array} \right\}$$

$$(5.72)$$

In order to evaluate the above integrals, their integrands are expressed in terms of the variable ξ

$$h_{\alpha}^{ij} = \int_{\Gamma_j} \psi_{\alpha} v_n \, ds = \int_{-1}^{1} \psi_{\alpha}(\xi) \, \frac{\cos \phi(\xi)}{2\pi r(\xi)} \, |J(\xi)| \, d\xi \qquad (\alpha = 1, 2, 3) \tag{5.73}$$

Similarly, we obtain

$$\int_{\Gamma_{j}} u_{n} v \, ds = \int_{\Gamma_{j}} (\psi_{1} u_{n}^{1} + \psi_{2} u_{n}^{2} + \psi_{3} u_{n}^{3}) v \, ds$$
$$= g_{1}^{ij} u_{n}^{1} + g_{2}^{ij} u_{n}^{2} + g_{3}^{ij} u_{n}^{3}$$
(5.74)

where

$$g_{\alpha}^{ij} = \int_{\Gamma_{J}} \psi_{\alpha} v \, ds$$

= $\int_{-1}^{1} \psi_{\alpha}(\xi) \frac{\ell n r(\xi)}{2\pi} |J(\xi)| d\xi. \quad (\alpha = 1, 2, 3)$ (5.75)

The computation of integrals (5.73) and (5.75) is performed numerically following a procedure similar to that employed for the linear element. Thus, for the outside integration $(i \neq j)$ the conventional Gauss integration is utilized. For the inside integration (i = j), the analytical method for the computation of g_{α}^{ij} becomes too

complicated and it is not recommended. Instead, the most suitable turns out to be the method of extracting the singularity. The elements H_{ii} are computed using the indirect method presented in Section 5.4.3, avoiding, thus, the evaluation of singular integrals.



Figure 5.13 Discontinuous parabolic element in global and local coordinate systems.

The corner points or the points where there is a change in the boundary conditions are treated using one-sided discontinuous parabolic elements (see Fig. 5.13 for $\kappa_2 = 1$). For discontinuous parabolic elements, the coefficients of the polynomial (5.59) are determined from the conditions

$$\begin{cases}
f(-\kappa_1) = f_1 \\
f(0) = f_2 \\
f(\kappa_2) = f_3
\end{cases}$$
(5.76)

which yield

$$\left. \begin{array}{c} \alpha_0 - \alpha_1 \,\kappa_1 + \alpha_2 \,\kappa_1^2 = f_1 \\ \alpha_0 = f_2 \\ \alpha_0 + \alpha_1 \,\kappa_2 + \alpha_2 \,\kappa_2^2 = f_3 \end{array} \right\}$$

$$(5.77)$$

Equations (5.77) are solved for the unknown coefficients α_0 , α_1 and α_2 giving

$$\alpha_{0} = f_{2}$$

$$\alpha_{1} = -\frac{\kappa_{2}^{2}}{K}f_{1} + \frac{\kappa_{2}^{2} - \kappa_{1}^{2}}{K}f_{2} + \frac{\kappa_{1}^{2}}{K}f_{3}$$

$$\alpha_{2} = \frac{\kappa_{2}}{K}f_{1} - \frac{\kappa_{1} + \kappa_{2}}{K}f_{2} + \frac{\kappa_{1}}{K}f_{3}$$

$$(5.78)$$

where

 $K = \kappa_1 \kappa_2 \left(\kappa_1 + \kappa_2 \right)$

Introducing the above values of the coefficients α_i in Eq. (5.59), the latter takes the familiar form

$$f(\xi) = \sum_{\alpha=1}^{3} \psi_{\alpha}(\xi) f_{\alpha} \qquad (\alpha = 1, 2, 3)$$
(5.79)

where $\psi_{\alpha}(\xi)$ are shape functions given as

$$\psi_{1}(\xi) = \frac{\kappa_{2}}{K}(-\kappa_{2} + \xi)\xi$$

$$\psi_{2}(\xi) = \frac{\kappa_{1} + \kappa_{2}}{K} \left[\kappa_{1}\kappa_{2} + (\kappa_{2} - \kappa_{1})\xi - \xi^{2}\right]$$

$$\psi_{3}(\xi) = \frac{\kappa_{1}}{K}(\kappa_{1} + \xi)\xi$$

$$(5.80)$$

Evidently, Eqs. (5.80) reduce to the shape functions of Eq. (5.65) for $\kappa_1 = \kappa_2 = 1$. Higher order elements can be derived by choosing interpolating polynomials of higher degree and following exactly the same procedure as for the quadratic element.

All continuous elements derived on the basis of the polynomial (5.55) for $n \ge 1$, produce a continuous variation of the boundary quantity on the whole boundary, that is, they do not exhibit jumps at the interelement nodes. This type of continuity is called C^0 continuity. These elements, however do not ensure continuity of the derivative at the interelement nodes. This continuity, referred to as C^1 continuity, can be achieved using shape functions described by special third order polynomials known as *Hermite polynomials* or *Hermite interpolation functions*. In this case, the element has two nodes which are placed at its extreme points. Unknowns are the values f_1 and f_2 of the boundary quantity, along with the corresponding derivatives $\theta_1 = (df/d\xi)_1$ and $\theta_2 = (df/d\xi)_2$. The boundary quantity is expressed as

$$f(\xi) = \psi_1(\xi) f_1 + \psi_2(\xi) f_2 + \psi_3(\xi) \theta_1 + \psi_4(\xi) \theta_2$$
(5.81)

in which the shape functions $\psi_{\alpha}(\xi)$ are given by the Hermite polynomials [1]

$$\psi_{1}(\xi) = \frac{1}{4}(1-\xi)^{2}(2+\xi), \qquad \psi_{2}(\xi) = \frac{1}{4}(1+\xi)^{2}(2-\xi)$$

$$\psi_{3}(\xi) = \frac{1}{4}(1-\xi)^{2}(1+\xi), \qquad \psi_{4}(\xi) = -\frac{1}{4}(1+\xi)^{2}(1-\xi)$$
(5.82)

5.6 Near-singular integrals

Once the unknown boundary quantities have been established from the solution of the boundary integral equations, the values of the potential u and its derivatives $u_{,x} = \partial u/\partial x$ and $u_{,y} = \partial u/\partial y$ at internal points can be computed using the expressions (4.11), (4.15) and (4.16), or the corresponding ones for the linear and the parabolic element approximation. The influence coefficients involved in the aforementioned equations are expressed in terms of line integrals on the elements Γ_j whose integrands involve factors of the form

$$\ell nr, \quad \frac{1}{r}, \quad \frac{1}{r^2} \tag{5.83}$$

where r = |P - q| is the distance between the points $P \in \Omega$ and $q \in \Gamma_j$. Clearly, because point P lies inside the domain Ω , while point q lies on the boundary, it is always $r \neq 0$. Therefore, these integrals, at least theoretically, are regular since the value of their integrands is always finite. When point P lies *far* from the boundary, the functions (5.83) have a smooth variation and consequently the conventional Gauss integration gives accurate results. However, when point P lies *near* the boundary, the functions (5.83) may take a very large, though finite, value and, thus, their variation is not smooth anymore.



Figure 5.14 Linear element and internal point P near the boundary, $d = \min(r) = 0.02 \ll 1$.

In order to illustrate this behavior, we consider the element of Fig. 5.14. The internal point P lies near the boundary. Applying Eqs. (5.3), one obtains the global coordinates in terms of the local coordinate ξ ,

$$x(\xi) = 2 - \xi$$
$$y(\xi) = 2 + \xi$$

and Eq. (5.12) gives the relative distance between points P and q as

$$r(\xi) = \left\{ \left[x(\xi) - 2.48 \right]^2 + \left[y(\xi) - 1.48 \right]^2 \right\}^{1/2}$$
$$= \left[(0.48 + \xi)^2 + (0.52 + \xi)^2 \right]^{1/2}$$
(5.84)

The variation of the functions $1/r(\xi)$ and $1/r(\xi)^2$ is shown in Figs. (5.15) and (5.16), respectively. We notice the large values of these functions at $\xi = -0.5$, that is at point A, which is the normal projection of point P on the element.



Figure 5.15 Variation of the function $1/r(\xi)$ along the element.

Consequently, the integrals of the functions (5.83) for points P near the boundary behave like singular integrals, although they are not. In the literature, these integrals are known as *near-singular integrals*. Their evaluation faces considerable difficulties, because neither the conventional Gauss integration nor those methods suitable for singular integrals can be employed. Nevertheless, other special techniques have been developed for their evaluation. Among them, the most popular are: the method of the *element subdivision*, and the method of the *coordinate transformation*. The first one has been discussed in detail by Lachat and Watson [14] and Kane et al. [15], while the second one has been presented by Telles [16]. Only the first method is presented below, while the other one can be found in the relevant literature.

The method of element subdivision

This method has been successfully employed to achieve a uniform accuracy for the values of the line integrals. The success of this technique is based on the user's experience. This technique is illustrated by evaluating the following integral



Figure 5.16 Variation of the function $1/r(\xi)^2$ along the element.

$$\int_{-1}^{1} \frac{1}{r(\xi)} d\xi \tag{5.85}$$

The plot of its integrand, which is depicted in Fig. 5.15, suggests subdivision of the interval [-1,+1] in four subintervals:

Subinterval 1: [-1,-0.6] Subinterval 2: [-0.6,-0.5] Subinterval 3: [-0.5,-0.4] Subinterval 4: [-0.4,+1]

In each subinterval, the integral can be evaluated using any integration rule such as the trapezoidal rule, Simpson's rule, Newton-Cotes integration formulae, etc. The Gauss integration is also recommended for this case. Its application requires first transformation of each subinterval onto the interval [-1,+1]. If ξ_k and ξ_{k+1} represent the end points of the k-th subinterval, then the linear transformation

$$\xi = \frac{\xi_{k+1} + \xi_k}{2} + \frac{\xi_{k+1} - \xi_k}{2}\eta \qquad (-1 \le \eta \le 1)$$
(5.86)

serves the purpose. The value of the integral over the k-th subinterval is given as

$$I_{k} = \frac{\xi_{k+1} - \xi_{k}}{2} \sum_{i=1}^{n} \frac{1}{r\left(\frac{\xi_{k+1} + \xi_{k}}{2} + \frac{\xi_{k+1} - \xi_{k}}{2}\eta_{i}\right)} w_{i}$$
(5.87)

where η_i and w_i (i = 1, 2, ..., n) are the abscissas and weights of the *n*-th order Gauss integration.

The integral (5.85) has been computed numerically using various subdivisions of the element shown in Fig. 5.14 and the results are given in Table 5.2 along with the exact value. These results reveal that the choice of element subdivision greatly affects the accuracy, even though the total number of integration points remains the same (20 points). Therefore, special care should be taken for the computation of near-singular integrals to avoid an uncontrolled error. A general rule for acceptable results is to choose two equal subintervals at both sides of point A, whose length is sufficiently smaller than the distance d (see Fig. 5.14).

Number of subintervals	Subintervals	Number of Gauss points	$\int_{-1}^1 \frac{1}{r} d\xi$
1	[-1.0, +1.0]	20	7.55905
2	[-1.0, -0.5]	10	6.34858
	[-0.5, +1.0]	10	
2	[-1.0, -0.5]	12	6.29535
	[-0.5, +1.0]	8	
2	[-1.0, -0.5]	8	6.35520
	[-0.5, +1.0]	12	
4	[-1.0, -0.6]	5	6.30650
	[-0.6, -0.5]	4	
	[-0.5, -0.4]	5	
	[-0.4, +1.0]	6	
4	[-1.0, -0.6]	6	6.30934
	[-0.6, -0.5]	4	
	[-0.5, -0.4]	4	
	[-0.4, +1.0]	6	
Exact value			6.309586

Table 5.2Values of the near-singular integral (5.85) for various
subdivisions of the element.

5.7 References

From the up to this point presentation of BEM, it becomes evident that its evolution as a computational method for solving realistic engineering problems is based on its success in solving singular boundary integral equations. Therefore, the technology of the boundary element, namely, the construction of various types of elements, the effective integration over them, especially of the singular kernels, as well as their manipulation in order to treat discontinuities of the boundary quantities, is among the most important ingredients of BEM and has been a field of intense research ever since BEM appeared as a computational method. For more information about this subject, the reader is advised to look in books by Brebbia and Dominguez [7], Kane [9], and Banerjee and Butterfield [13]. Regarding the evaluation of singular integrals, a rich technical literature is available. Suggestively, we mention the work of Hall [17], Doblare [18] as well as the recently published book by Sladek and Sladek [19], who presented methods for the evaluation of singular integrals. Hayami and Brebbia [20], apart from the singular integrals, treated the near-singular ones. A method for the computation of line integrals with logarithmic singularity has also been presented by Katsikadelis and Armenakas [21]. Theocaris and his co-workers [22, 23] have published extended work on the integration of singular integrals. Considerable work, especially on the evaluation of hypersingular integrals has been published by Guiggiani and his co-workers [24, 25]. The reader can also find extended literature related to this subject in the chapters Computational Aspects of the proceedings of the International Boundary Element Conferences (Computational Mechanics Publications, Southampton), which have taken place uninterruptedly for the last 22 years.

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Problems

- **5.1.** Derive the shape functions for the cubic element when the interior nodal points are placed at
 - (i) $\xi_2 = -1/3$ and $\xi_3 = 1/3$,
 - (ii) $\xi_2 = -1/2$ and $\xi_3 = 1/2$.
- **5.2.** Given the circular sector of radius R = 3 and angle $\theta_0 = \pi/12$, compute its area by approximating the circular arc with (i) linear, (ii) quadratic and (iii) cubic elements. For each case determine also the error.
- **5.3.** Compute the near-singular integral using Gauss integration

$$I = \int_{-1}^{+1} \frac{dx}{\left[(x - 0.25)^2 + 0.05\right]^4}$$

5.4. Compute the integral

$$\int_{\Gamma_J} \psi_1(\xi) \, \ell nr \, ds$$

when Γ_j is a quadratic element passing through the points 1(4.30, 2.50), 2(4.10, 2.90), 3(3.80, 3.20) and the source lies at point P (4.15, 2.65).

5.5. Compute the integrals $g_{\alpha}{}^{ij}$ ($\alpha = 1, 2$) for the linear element with nodal points $\mathbf{1}(1, 2)$ and $\mathbf{2}(1.5, 2.3)$, when $\kappa_1 = \kappa_2 = 0.5$ and the source lies consecutively at point 1 and point 2.

Chapter 6

Applications

6.1 Introduction

As it was mentioned in Chapter 3, the Laplace and Poisson equations describe the behavior of many physical systems. In this chapter the BEM will be employed to solve several problems, such as torsion of non-circular prismatic bars, deflection of membranes, bending of simply supported plates, heat conduction and fluid flow. All these problems are governed either by the Laplace or the Poisson equation.

6.2 Torsion of non-circular bars

6.2.1 The warping function

Let us consider a bar of arbitrary cross-section twisted by moments M_t applied at its ends (Fig. 6.1). The cross-section is constant along the length of the bar. According to Saint-Venant's torsion theory [1, 2], the deformation of the bar consists of (a) rotations of the cross-sections about an axis passing through the *twist center* of the bar, and (b) warping of the cross-sections, which is the same for all sections. Choosing the origin of the coordinate system at the twist center of an end section (Fig. 6.1), the rotation at a distance z is θz , where θ is a constant expressing the rotation of a cross-section per unit length. Referring to Fig. 6.2 and assuming that this rotation is small, the displacements u and v of point A(x, y, z) due to the rotation are determined as

$$u = -(AA')\sin\alpha = -r\,\theta\,z\frac{y}{r} = -\theta\,z\,y \tag{6.1a}$$

$$v = (AA')\cos\alpha = r \theta z \frac{x}{r} = \theta z x$$
 (6.1b)

The warping of the cross-section is defined as



Figure 6.1 Bar of arbitrary cross-section twisted by moments end M_t .



Figure 6.2 Displacement components in a cross-section of a twisted bar.

$$w = \theta \phi(x, y) \tag{6.2}$$

where $\phi(x, y)$ is the warping function.

The displacements given by Eqs. (6.1) and (6.2) yield the following strain components

$$\varepsilon_{x} = \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{y} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{z} = \frac{\partial w}{\partial z} = 0, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta \left(\frac{\partial \phi}{\partial x} - y \right)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left(\frac{\partial \phi}{\partial y} + x \right)$$
(6.3)

Hence, for homogeneous linear elastic material the corresponding stress components resulting from the above strain components are

$$\sigma_{x} = \sigma_{y} = \sigma_{z} = \tau_{xy} = 0$$

$$\tau_{xz} = G\theta \left(\frac{\partial\phi}{\partial x} - y\right)$$

$$\tau_{yz} = G\theta \left(\frac{\partial\phi}{\partial y} + x\right)$$
(6.4)

The equilibrium equations for the three-dimensional state of stress in the absence of body forces, are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0$$
(6.5a)

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$
(6.5b)

$$\frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$
(6.5c)

Introducing the constitutive relations (6.4) into Eqs. (6.5) we obtain

$$\frac{\partial \tau_{zz}}{\partial z} = 0 \tag{6.6a}$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \tag{6.6b}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{6.6c}$$

The first two of Eqs. (6.6) always hold, since the stress components τ_{xz} and τ_{yz} are independent of z. The last one, Eq. (6.6c), expresses the condition which must be satisfied by the warping function $\phi(x, y)$.

In addition, the stress components should satisfy the traction boundary conditions on the surface of the bar, namely

$$\sigma_{x} n_{x} + \tau_{xy} n_{y} + \tau_{xz} n_{z} = t_{x}$$

$$\tau_{xy} n_{x} + \sigma_{y} n_{y} + \tau_{yz} n_{z} = t_{y}$$

$$\tau_{xz} n_{x} + \tau_{yz} n_{y} + \sigma_{z} n_{z} = t_{z}$$

$$(6.7)$$

where n_x , n_y , n_z are the direction cosines of the outward normal vector and t_x , t_x , t_z are the traction components on the surface of the bar.

(a) We examine first the boundary conditions on the cylindrical surface of the bar. This surface is traction free, that is, $t_x = t_y = t_z = 0$. Moreover, it is $n_z = 0$. Taking into account Eqs. (6.4), we can readily show that the first two of the boundary conditions (6.7) are identically satisfied, while the third one yields

$$\left(\frac{\partial\phi}{\partial x} - y\right)n_x + \left(\frac{\partial\phi}{\partial y} + x\right)n_y = 0$$

which may also be written as

$$rac{\partial \phi}{\partial x} n_x + rac{\partial \phi}{\partial y} n_y = y n_x - x n_y$$

or

$$\frac{\partial \phi}{\partial n} = y \, n_x - x \, n_y \tag{6.8}$$

The warping function ϕ may be determined from Eqs. (6.6c) and (6.8) by solving a Neumann problem for the Laplace equation, provided that the function

$$\phi_n(s) = y \, n_x - x \, n_y \tag{6.9}$$

satisfies the existence condition for the solution of the Neumann problem, that is

$$\int_{\Gamma} \phi_n \, ds = 0 \tag{6.10}$$

This condition results from Green's identity, Eq. (2.16) for v = 1 and $u = \phi$. Indeed, using Eqs. (2.3), Eq. (6.9) is written as

$$egin{aligned} \phi_n &= y rac{dy}{ds} + x rac{dx}{ds} \ &= rac{1}{2} rac{d}{ds} ig(x^2 + y^2ig) \end{aligned}$$

Consequently, noting that the function $\frac{1}{2}(x^2 + y^2)$ is continuous on the whole boundary, we obtain

$$\int_{\Gamma} \phi_n \, ds = \frac{1}{2} \int_{\Gamma} \frac{d}{ds} \left(x^2 + y^2 \right) ds$$
$$= \frac{1}{2} \left[x^2 + y^2 \right]_B^B = 0$$

where B is any point on the boundary taken as the origin for the variable s.

(b) On the end cross-sections z = 0 and z = L, it is $n_x = n_y = 0$ and $n_z = 1$. Thus, the boundary conditions (6.7) become

$$\tau_{zz} = t_z, \quad \tau_{yz} = t_y, \quad \sigma_z = t_z = 0$$
 (6.11)

which state that the end cross-sections are subjected only to tangential tractions. We can readily prove that the stress resultants of these tractions vanish. Namely,

$$\int_{\Omega} \tau_{xz} \, d\Omega = 0 \quad \text{and} \quad \int_{\Omega} \tau_{yz} \, d\Omega = 0 \tag{6.12}$$

Indeed, the third of Eqs. (6.5) becomes

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{6.13}$$

Moreover, the first of Eqs. (6.12) may be written as

$$\begin{split} \int_{\Omega} \tau_{zz} \, d\Omega &= \int_{\Omega} \bigg[\tau_{zz} + x \bigg(\frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \bigg) \bigg] d\Omega \\ &= \int_{\Omega} \bigg[\frac{\partial (x \tau_{zz})}{\partial x} + \frac{\partial (x \tau_{yz})}{\partial y} \bigg] d\Omega \end{split}$$

Next applying Gauss divergence theorem (2.9) and using the last of Eqs. (6.7), the above equation yields

$$\int_{\Omega} \tau_{xz} \, d\Omega = \int_{\Gamma} x \left(\tau_{xz} \, n_x + \tau_{yz} \, n_y \right) ds = 0 \tag{6.14}$$

since $n_z = 0$ and $t_z = 0$.

In a similar way, we can prove the second of Eqs. (6.12).

The moment resultant on the cross-section z = 0 is going to be

$$M_{t} = \int_{\Omega} (x \tau_{yz} - y \tau_{xz}) d\Omega = G\theta \int_{\Omega} \left(x^{2} + y^{2} + x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) d\Omega$$
(6.15)

Setting

$$I_{t} = \int_{\Omega} \left(x^{2} + y^{2} + x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) d\Omega$$
(6.16)

we arrive at

$$M_t = GI_t \theta \tag{6.17}$$

The constant quantity I_t , which depends only on the shape of the cross-section, is usually referred to as *torsional constant*. The quantity GI_t is called the *torsional rigidity* of the cross-section. Denoting by $\overline{\theta} = \theta L$ the relative rotation of the end cross-sections, Eq. (6.17) may also be written as

$$M_t = \frac{GI_t}{L}\overline{\theta} \tag{6.18}$$

The quantity GI_t/L expresses the torsional stiffness coefficient of a bar having length L. This coefficient appears in the stiffness matrix of grid elements or three dimensional beam elements.

From the foregoing analysis, we conclude that the determination of the torsional constant of bars as well as of the shear stresses due to torsion, require the establishment of the warping function ϕ of the cross-section. For simple cross-sectional geometries, (e.g. elliptical, rectangular, triangular) the warping function ϕ can be determined using exact or approximate analytical solutions. However, for cross-sections of complex shape, as it happens with realistic engineering problems, it is necessary to solve a Neumann problem for the Laplace equation in an arbitrary domain Ω . Hence, the warping function ϕ is established as the solution of the following boundary value problem

$$\nabla^2 \phi = 0 \quad \text{in } \Omega \tag{6.19a}$$

$$\frac{\partial \phi}{\partial n} = y n_x - x n_y \quad \text{on} \quad \Gamma$$
 (6.19b)

The function ϕ is determined exactly apart from an arbitrary constant term. That is, a function ϕ^* is obtained in the form

$$\phi^{*}(x,y) = \phi(x,y) + C \tag{6.20}$$

where $\phi(x, y)$ is the exact solution and C is a constant.

Apparently, the stress components and the torsional constant are not influenced by this arbitrary constant, because, according to Eqs. (6.4) and (6.16), only the derivatives of ϕ are required for the evaluation of these quantities. With respect to the displacement w, the arbitrary constant introduces a rigid body motion in the direction of the bar axis (see Eq. (6.2)), which, however, does not influence the deformation of the cross-section. The constant C can be determined by specifying the displacement w at a point of the cross-section, e.g. w = 0, which yields $\phi^* = 0$ at this point and then $\phi(x, y) = \phi^*(x, y) - C$. It is advisable to choose the *twist center* of the cross-section as the point of zero axial displacement. Therefore, the establishment of this point should precede, if it not a priori known.

Determination of the twist center

The solution of the boundary value problem (6.19) gives the warping surface, if the origin of the coordinate axes is taken at the twist center of the cross-section, that is the point which does not undergo any displacement during the rotation of the cross-section. In axisymmetric cross-sections (e.g. rectangle, equilateral triangle, ellipse, etc) the twist center can be readily established by inspection since it coincides with the geometric center of the cross-section. However, for cross-sections of arbitrary shape, the twist center is not known and it should be determined in order to establish the warping function. This can be achieved by working as follows.

When the origin O does not coincide with the twist center (x_0, y_0) , Eqs. (6.1a,b), (6.2), (6.3), (6.4), (6.6c) and (6.8) are written respectively as

$$u = -\theta \, z \left(y - y_0 \right) \tag{6.21a}$$

$$v = \theta z (x - x_0) \tag{6.21b}$$

$$w = \theta \phi(x, y) \tag{6.21c}$$

$$\varepsilon_{x} = 0, \quad \varepsilon_{y} = 0, \quad \varepsilon_{z} = 0, \quad \gamma_{xy} = 0$$

$$\gamma_{xz} = \theta \left[\frac{\partial \phi}{\partial x} - (y - y_{0}) \right]$$

$$\gamma_{yz} = \theta \left[\frac{\partial \phi}{\partial y} + (x - x_{0}) \right]$$
(6.22)

$$\sigma_{x} = \sigma_{y} = \sigma_{z} = \tau_{xy} = 0$$

$$\tau_{xz} = G\theta \left[\frac{\partial \phi}{\partial x} - (y - y_{0}) \right]$$

$$\tau_{yz} = G\theta \left[\frac{\partial \phi}{\partial y} + (x - x_{0}) \right]$$
(6.23)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{6.24}$$

$$\frac{\partial \phi}{\partial n} = (y - y_0) n_x - (x - x_0) n_y \tag{6.25}$$

Equation (6.25) may further be written as

$$\frac{\partial}{\partial n}(\phi + y_0 x - x_0 y) = y n_x - x n_y \tag{6.26}$$

or

$$\frac{\partial \phi^*}{\partial n} = yn_x - xn_y \tag{6.27}$$

where

$$\phi^* = \phi + y_0 x - x_0 y + C \tag{6.28a}$$

and consequently

$$\phi = \phi^* - y_0 x + x_0 y - C \tag{6.28b}$$

Since $\nabla^2 \phi^* = \nabla^2 \phi$, it is apparent that the Neumann problem being solved, is actually

which yields the function ϕ^* . As a result, the stress components and the torsional constant should be expressed in terms of the function ϕ^* and not ϕ . This can be accomplished by introducing ϕ from Eq. (6.28b) into the expressions (6.23), which gives

$$\tau_{xz} = G\theta \left(\frac{\partial \phi^*}{\partial x} - y \right) \tag{6.30a}$$

$$\tau_{yz} = G\theta \left(\frac{\partial \phi^*}{\partial y} + x \right) \tag{6.30b}$$

Using the above relations in Eq. (6.15) to express the moment M_t with respect to the origin of the coordinate axes, we obtain

$$I_t = \int_{\Omega} \left(x^2 + y^2 + x \frac{\partial \phi^*}{\partial y} - y \frac{\partial \phi^*}{\partial x} \right) d\Omega$$
(6.31)

On the basis of Eqs. (6.28a), (6.29), (6.30) and (6.31), it can be concluded that.

- (a) If the origin of the coordinates does not coincide with the twist center of the cross-section, then the warping function ϕ^* , obtained as the solution of the Neumann problem, has undergone a rigid body rotation in the plane of the cross-section and a displacement parallel to the axis of the bar (see Ref. [1]).
- (b) The stress components τ_{xz} , τ_{yz} , and the torsion constant I_t do not depend on the position of the origin of the coordinate axes.

The actual warping function ϕ is determined by establishing first the quantities x_0 , y_0 and C. These quantities can be obtained from the minimization of the strain energy produced by axial normal warping stresses, which are ignored by the Saint-Venant's theory. This energy is given as [5]

$$\Pi(x_0, y_0, C) = \frac{1}{2} E\theta^2 \int_{\Omega} \phi^2 d\Omega$$

= $\frac{1}{2} E\theta^2 \int_{\Omega} (\phi^* - y_0 x + x_0 y - C)^2 d\Omega$ (6.32)

The minimization conditions require

$$\frac{\partial \Pi}{\partial x_0} = 0$$

$$\frac{\partial \Pi}{\partial y_0} = 0$$

$$\frac{\partial \Pi}{\partial C} = 0$$
(6.33)

Equation (6.32) is differentiated with respect to each of the three quantities to yield

$$\left. \begin{cases}
 I_x x_0 - I_{xy} y_0 - S_x C = -I_1 \\
 I_{xy} x_0 - I_y y_0 - S_y C = -I_2 \\
 S_x x_0 - S_y y_0 - AC = -I_3
 \end{cases} \right\}
 \tag{6.34}$$

where it was set

$$A = \int_{\Omega} d\Omega, \qquad S_{x} = \int_{\Omega} y \, d\Omega, \qquad S_{y} = \int_{\Omega} x \, d\Omega$$

$$I_{x} = \int_{\Omega} y^{2} \, d\Omega, \qquad I_{xy} = \int_{\Omega} xy \, d\Omega, \qquad I_{y} = \int_{\Omega} x^{2} \, d\Omega$$

$$I_{1} = \int_{\Omega} y \, \phi^{*} \, d\Omega, \qquad I_{2} = \int_{\Omega} x \, \phi^{*} \, d\Omega, \qquad I_{3} = \int_{\Omega} \phi^{*} \, d\Omega$$

$$(6.35)$$

Since the torsion problem is solved by the BEM, the domain integrals in Eqs. (6.35) should be converted to boundary line integrals in order to maintain the pure boundary character of the method. This can be achieved using Eqs. (2.5), (2.6) and (2.9). Thus we can write

$$A = \int_{\Omega} d\Omega = \frac{1}{2} \int_{\Omega} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) d\Omega = \frac{1}{2} \int_{\Gamma} (xn_x + yn_y) ds$$
(6.36)

$$S_{x} = \int_{\Omega} y \, d\Omega = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial y} (y^{2}) \, d\Omega = \frac{1}{2} \int_{\Gamma} y^{2} n_{y} \, ds \tag{6.37}$$

$$S_y = \int_{\Omega} x \, d\Omega = \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial x} (x^2) \, d\Omega = \frac{1}{2} \int_{\Gamma} x^2 n_x \, ds \tag{6.38}$$

$$I_{x} = \int_{\Omega} y^{2} d\Omega = \frac{1}{3} \int_{\Omega} \frac{\partial}{\partial y} (y^{3}) d\Omega = \frac{1}{3} \int_{\Gamma} y^{3} n_{y} ds$$
(6.39)

$$I_y = \int_{\Omega} x^2 \, d\Omega = \frac{1}{3} \int_{\Omega} \frac{\partial}{\partial x} (x^3) \, d\Omega = \frac{1}{3} \int_{\Gamma} x^3 n_x \, ds \tag{6.40}$$

$$I_{xy} = \int_{\Omega} xy \, d\Omega = \frac{1}{4} \int_{\Omega} \left[\frac{\partial}{\partial x} (x^2 y) + \frac{\partial}{\partial y} (xy^2) \right] d\Omega$$
$$= \frac{1}{4} \int_{\Gamma} xy (xn_x + yn_y) \, ds \tag{6.41}$$

To treat the integrals involving the function ϕ^* , Green's second identity (2.16) is applied consecutively for the functions

$$u_1 = \frac{yr^2}{8}, \quad u_2 = \frac{xr^2}{8}, \quad u_3 = \frac{r^2}{4} \qquad (r^2 = x^2 + y^2)$$
 (6.42)

and for $v = \phi^*$. Noting that $abla^2 u_1 = y$, $abla^2 u_2 = x$ and $abla^2 u_3 = 1$, we obtain

$$I_{1} = \int_{\Omega} y \phi^{*} d\Omega = \int_{\Gamma} \left(\phi^{*} \frac{\partial u_{1}}{\partial n} - u_{1} \frac{\partial \phi^{*}}{\partial n} \right) ds$$
(6.43)

$$I_2 = \int_{\Omega} x \phi^* d\Omega = \int_{\Gamma} \left(\phi^* \frac{\partial u_2}{\partial n} - u_2 \frac{\partial \phi^*}{\partial n} \right) ds$$
(6.44)

$$I_{3} = \int_{\Omega} \phi^{*} d\Omega = \int_{\Gamma} \left(\phi^{*} \frac{\partial u_{3}}{\partial n} - u_{3} \frac{\partial \phi^{*}}{\partial n} \right) ds$$
(6.45)

The foregoing integrals are computed using BEM with constant elements.

The steps required in order to solve the torsion problem for bars with arbitrary cross-section can be summarized as:

- (a) The function ϕ^* is determined as the solution of the Neumann problem described by Eqs. (6.29) with respect to the arbitrarily chosen xy-system of axes. Special care should be taken to ensure the existence of a solution for the Neumann problem (see Example 4.2).
- (b) The coordinates x_0 , y_0 of the twist center and the constant C are computed from the solution of Eqs. (6.34).
- (c) The warping function ϕ is evaluated using Eq. (6.28b).
- (d) The boundary stress τ_{tz} is computed using the procedure described in the following Section 6.2.2 and the torsion constant I_t is given by Eq. (6.31) or even better by its boundary integral form (6.47), which is derived right below.

The domain integral (6.31) is converted into a boundary line integral to avoid not only the domain integration but also the evaluation of the derivatives of ϕ^* , which are involved in the integrand. This is achieved following the procedure below.

Equation (6.31) may be written as

$$I_{t} = \int_{\Omega} \left[\frac{\partial}{\partial x} \left(xy^{2} - y\phi^{*} \right) + \frac{\partial}{\partial y} \left(yx^{2} + x\phi^{*} \right) \right] d\Omega$$
(6.46)

Using the Gauss divergence theorem (2.9) the above integral is transformed to the following boundary line integral

$$I_t = \int_{\Gamma} \left[\left(xy^2 - y\phi^* \right) n_x + \left(yx^2 + x\phi^* \right) n_y \right] ds$$
(6.47)

Remarks

The solution of the Saint-Venant torsion problem can also be obtained by determining the following functions:

(a) The function $\psi(x,y)$, which is the conjugate of $\phi(x,y)$ and the solution of the Dirichlet problem

$$abla^2 \psi = 0 \quad \text{in } \Omega$$
 $\psi = \frac{1}{2} (x^2 + y^2) + C \quad \text{on } \Gamma$

where C is an arbitrary constant. In this case, the shear stresses and the torsional constant are expressed as

$$\tau_{xz} = G\theta \left(\frac{\partial\psi}{\partial y} - y\right)$$

$$\tau_{yz} = G\theta \left(\frac{\partial\psi}{\partial x} + x\right)$$

$$I_t = \int_{\Omega} \left(x^2 + y^2 - x\frac{\partial\psi}{\partial x} - y\frac{\partial\psi}{\partial y}\right) d\Omega$$

(b) The Prandtl's stress function F(x,y) which is the solution of the Dirichlet problem

$$abla^2 F = -2 \quad \text{in } \Omega$$
 $F = C \quad \text{on } \Gamma$

where C is an arbitrary constant. The shear stresses and the torsional constant are given as

$$\tau_{xz} = G\theta \frac{\partial F}{\partial y}$$

$$\tau_{yz} = -G\theta \frac{\partial F}{\partial x}$$

$$I_t = -\int_{\Omega} \left(x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \right) d\Omega$$

The formulation in terms of the warping function ϕ is preferred over the formulations in terms of the foregoing two functions, because:

- (i) Once the function ψ or F is established, the determination of the warping function, hence of the axial displacements, from these functions requires the solution of an additional potential problem which is equally difficult as the original one.
- (ii) When the cross-section does not have holes, that is the domain Ω is simply connected, the arbitrary constant C appearing in the boundary condition can be given an arbitrary value, e.g. C = 0. However, when the cross-section has holes (hollow cross-section), that is the domain Ω is multiply connected, then the constant C does not, generally, take the same value on all the contours and its exact values should be determined. This can be done by imposing additional conditions which ensure the uniqueness of the displacements. Therefore, the solution of the problem becomes even more complicated.

6.2.2 Evaluation of stresses

The stress components τ_{xz} and τ_{yz} are evaluated using Eqs. (6.4). Apparently, these are determined by evaluating first the derivatives of the function ϕ . For points inside the domain of the cross-section, the derivatives can be computed by applying Eqs. (4.15) and (4.16).

The maximum values of the stresses appear on the boundary. The stress τ_{nz} is zero, whereas the stress τ_{tz} is given by the relation

$$\tau_{tz} = -\tau_{xz} \, n_y + \tau_{yz} \, n_x$$

which by virtue of Eqs. (6.4) becomes

$$\tau_{tz} = G\theta\left(\frac{\partial\phi}{\partial t} + x n_x + y n_y\right) \tag{6.48}$$

The derivative $\partial \phi / \partial t = \partial \phi / \partial s$ can be computed through numerical differentiation of ϕ along the boundary as it is described next.

Let us consider three consecutive nodal points i-1, i, i+1 on the boundary. The values ϕ_{i-1} and ϕ_{i+1} can be expressed in terms of the value of ϕ and its derivatives at point *i* using Taylor series expansions

$$\phi_{i-1} = \phi_i - (\phi_{,s})_i \, s_1 + \frac{1}{2} (\phi_{,ss})_i \, s_1^2 - \frac{1}{6} (\phi_{,sss})_i \, s_1^3 + \dots$$

$$\phi_{i+1} = \phi_i + (\phi_{,s})_i \, s_2 + \frac{1}{2} (\phi_{,ss})_i \, s_2^2 + \frac{1}{6} (\phi_{,sss})_i \, s_2^3 + \dots$$

where

$$s_1 = \frac{\ell_{i-1} + \ell_i}{2}$$
 and $s_2 = \frac{\ell_i + \ell_{i+1}}{2}$

with ℓ_i being the length of the *i*-th element.

In the above expressions, we neglect terms of order higher than the second and subsequently we eliminate the second derivatives. This yields the *central difference* approximation for the first derivative

$$(\phi_{,s})_i = \left(\frac{\partial\phi}{\partial t}\right)_i = \alpha_1 \phi_{i-1} + \alpha_2 \phi_i + \alpha_3 \phi_{i+1}$$
(6.49)

where

$$\alpha_1 = -\frac{s_2}{s_1(s_1 + s_2)}, \quad \alpha_2 = -\frac{s_1 - s_2}{s_1 s_2}, \quad \alpha_3 = \frac{s_1}{s_2(s_1 + s_2)}$$
(6.50)

If $s_1 = s_2 = \Delta s$, we obtain the known expression

$$\left(\frac{\partial\phi}{\partial t}\right)_{i} = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta s} \tag{6.51}$$

At points near the corners the derivative $\partial \phi / \partial t$ is discontinuous. For this reason *forward* (*backward*) *differences* should be employed when the corner precedes (follows) the point *i*. The finite difference expressions for the evaluation of $\partial \phi / \partial t$ for forward and backward differences will be derived next.

For forward differences we consider the values ϕ_i , ϕ_{i+1} and ϕ_{i+2} . The Taylor series approximations of ϕ_{i+1} and ϕ_{i+2} in terms of the value of ϕ and its derivatives at point *i* give

$$\phi_{i+1} = \phi_i + (\phi_{,s})_i \, s_1 + \frac{1}{2} (\phi_{,ss})_i \, s_1^2 + \frac{1}{6} (\phi_{,sss})_i \, s_1^3 + \dots$$

$$\phi_{i+2} = \phi_i + (\phi_{,s})_i \, (s_1 + s_2) + \frac{1}{2} (\phi_{,ss})_i \, (s_1 + s_2)^2 + \frac{1}{6} (\phi_{,sss})_i \, (s_1 + s_2)^3 + \dots$$

Neglecting the terms of order higher than the second, the above equations yield

$$(\phi_s)_i = \left(\frac{\partial\phi}{\partial t}\right)_i = \alpha_1 \phi_i + \alpha_2 \phi_{i+1} + \alpha_3 \phi_{i+2}$$
(6.52)

where

$$\alpha_{1} = -\frac{2s_{1} + s_{2}}{s_{1}(s_{1} + s_{2})}, \quad \alpha_{2} = \frac{s_{1} + s_{2}}{s_{1}s_{2}}, \quad \alpha_{3} = -\frac{s_{1}}{s_{2}(s_{1} + s_{2})}$$

$$s_{1} = \frac{\ell_{i} + \ell_{i+1}}{2} \quad \text{and} \quad s_{2} = \frac{\ell_{i+1} + \ell_{i+2}}{2}$$
(6.53)

For backward differences we consider the values ϕ_{i-2} , ϕ_{i-1} , ϕ_i and following an analogous procedure, we obtain

$$(\phi_{,s})_i = \left(\frac{\partial\phi}{\partial t}\right)_i = \alpha_1 \phi_i + \alpha_2 \phi_{i-1} + \alpha_3 \phi_{i-2}$$
(6.54)

$$\alpha_1 = \frac{2s_1 + s_2}{s_1(s_1 + s_2)}, \quad \alpha_2 = -\frac{s_1 + s_2}{s_1 s_2}, \quad \alpha_3 = \frac{s_1}{s_2(s_1 + s_2)}$$
(6.55)

$$s_1 = \frac{\ell_i + \ell_{i-1}}{2}$$
 and $s_2 = \frac{\ell_{i-1} + \ell_{i-2}}{2}$

6.2.3 Program TORSCON for solving the torsion problem with constant elements

The program LABECON can be readily modified to solve the torsion problem. Besides the main program, the subroutines that are modified are: INPUT, UINTER and OUTPUT. Moreover, three new subroutines have been added, namely, TORCENTER, TORSTIF and TORSTRESS, which compute the center of twist of the cross-section, the torsion constant I_t , and the boundary stress τ_{tz} , respectively. The listings of the main program and the modified subroutines as well as the new subroutines are given below.

```
С
     PROGRAM TORSCON
С
С
C This program solves the Saint-Venant TORSION problem
С
  as a Neumann problem for the Laplace equation using
С
  the boundary element method with (CON) stant
C boundary elements
C
     IMPLICIT REAL*8 (A-H,O-Z)
     CHARACTER*15 INPUTFILE, OUTPUTFILE
С
С
  Set the maximum dimensions
С
     PARAMETER (N=20)
     PARAMETER (IN=13)
С
C N= Number of boundary elelments equal to number of boundary
C
     nodes
C IN= Number of internal points where the function u is calculated
С
     DIMENSION INDEX(N)
     DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), G(N,N), H(N,N), UB(N)
     DIMENSION A(N,N), UNB(N), XIN(IN+1), YIN(IN+1), UIN(IN+1), SL(N), TTZ(N)
     DIMENSION AA(3,3), BB(3)
С
С
  Read the names and open the input and output files
С
     WRITE (*, '(A)')' Name of the INPUTFILE (max.15 characters)'
     READ (*, '(A)') INPUTFILE
     WRITE (*,'(A)')' Name of the OUTPUTFILE (max.15 characters)'
     READ (*,'(A)') OUTPUTFILE
     OPEN (1, FILE=INPUTFILE)
     OPEN (2, FILE=OUTPUTFILE)
```

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```
Read data from INPUTFILE
C
С
      CALL INPUT(XL,YL,XIN,YIN, INDEX, UB, N, IN)
С
  Compute the G matrix
С
С
      CALL GMATR(XL,YL,XM,YM,G,N)
С
  Compute the H matrix
С
С
      CALL HMATR(XL,YL,XM,YM,H,N)
С
  Form the system of equations AX=B
C
C
      CALL ABMATR(G, H, A, UNB, UB, INDEX, N)
C
  Solve the system of equations
С
С
      CALL SOLVEQ(A, UNB, N, LSING)
С
  Form the vectors U and UN of all the bounday values
С
С
      CALL REORDER (UB, UNB, INDEX, N)
С
  Compute the coordinates of the twist center of the
С
  cross-section and the Neumann's problem constant
С
C
      CALL TORCENTER (XL, YL, N, UB, UNB, AA, BB, XTC, YTC, CT)
C
С
  Compute the values of U at the internal points
¢
      CALL UINTER (XL, YL, XM, YM, XIN, YIN, UB, UNB, UIN, N, IN, XTC, YTC, CT)
С
С
   Compute the the torsion constant D
С
      CALL TORSTIF(XL, YL, N, UB, XTC, YTC, D)
С
  Compute the boundary stress Ttz
С
C
      CALL TORSTRESS (XL, YL, XM, YM, UB, XTC, YTC, TTZ, SL, N)
C
  Print the results in the OUTPUTFILE
С
Ç
      CALL OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, D, TTZ, N, IN, XTC, YTC, CT)
С
   Close input and output files
С
с
      CLOSE(1)
      CLOSE(2)
      STOP
      END
С
С
        C=:
C
      SUBROUTINE INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN)
С
С
   This subroutine reads the data from the input file
С
С
   and writes them in the output file
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*80 NAME, TITLE
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), INDEX(N), UB(N)
C
```

```
WRITE(2,100)
  100 FORMAT('',69('*'))
С
С
   Read user's name
С
      READ(1, '(A)')NAME
С
      WRITE(2, '(A) ')NAME
С
С
   Read the title of the program
С
      READ(1, '(A) ')TITLE
С
      WRITE(2, '(A) ') TITLE
С
      WRITE(2,200)N, IN
  200 FORMAT(//'BASIC PARAMETERS'//2X, 'NUMBER OF BOUNDARY ELEMENTS='
     1,13/2X, 'NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED
     1=',I3)
С
С
  Read the coordinates XL, YL of the extreme points of the boundary elements
С
      READ(1,*) (XL(I),YL(I),I=1,N)
С
С
   Write the coordinates in the output file
С
      WRITE(2,300)
  300 FORMAT (//2X, 'COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELE
     1MENTS',//2X, 'POINT', 9X, 'XL', 15X, 'YL')
      DO 20 I=1.N
   20 WRITE(2,400) I,XL(I),YL(I)
  400 FORMAT(2X, I3, 2(3X, E14.5))
С
С
   Compute the boundary values of Un and store in UB(I) (I=1,N-1),UB(N)=0.
с
      DO 10 I=1,N-1
      DX = XL(I+1) - XL(I)
      DY=YL(I+1)-YL(I)
      SL=DSQRT(DX**2+DY**2)
      ENX=DY/SL
      ENY=-DX/SL
      XM = (XL(I) + XL(I+1))/2.
      YM = (YL(I) + YL(I+1))/2.
      UB(I) \simeq YM * ENX - XM * ENY
   10 INDEX(I) = 1
      UB(N) = 0.
      INDEX (N) = 0
С
С
   Write the boundary conditions in the output file
C
      WRITE(2,500)
  500 FORMAT(//2X,'BOUNDARY CONDITIONS'//2X,'NODE',6X,'INDEX',
     1 7X, 'PRESCRIBED VALUE')
      DO 30 I=1,N
   30 WRITE(2,600) I, INDEX(I), UB(I)
  600 FORMAT (2X,13,9X,11,8X,E14.5)
C
С
   Read the coordinates of the internal points
С
      READ(1,*) (XIN(I),YIN(I),I=1,IN)
      RETURN
      END
С
C
C------
```

```
С
      SUBROUTINE TORCENTER (XL, YL, N, UB, UNB, AA, BB, XTC, YTC, CT)
С
С
С
   This subroutine computes the coordinates of the twist
С
   center of the cross-section and the arbitrary constant
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), UB(N), UNB(N)
      DIMENSION XC(4),YC(4),XI(4),WG(4)
      DIMENSION AA(3,3), BB(3)
С
С
         N = Number of boundary elements
С
     XL, YL = Coordinates of the extreme points of the
¢
               boundary elements
С
   XTC, YTC = Coordinates of the twist center
С
        CT = Arbitrary constant of the Neumann problem
С
        WG = Weights of the Gauss integration
С
        XI = Coordinates of the Gauss integration points
С
               in the interval [-1,1]
с
     XC, YC = Global coordinates of the Gauss integration points
С
      DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      PI=ACOS(-1.)
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
с
      AREA=0.
      SX=0.
      SY=0.
      AIX=0.
      AIY=0.
      AIXY=0.
      AI1=0.
      AI2=0.
      AI3=0.
С
      DO 10 I=1,N
      AX = (XL(I+1) - XL(I))/2.
      AY = (YL(I+1) - YL(I)) / 2.
      BX = (XL(I+1) + XL(I))/2.
      BY = (YL(I+1) + YL(I)) / 2.
      SL=DSQRT (AX**2+AY**2)
      ENX=AY/SL
      ENY=-AX/SL
      TERMA=0.
      TERMSX=0.
      TERMSY=0.
      TERMIX=0.
      TERMIY=0.
      TERMIXY=0.
      TERMI1=0.
      TERMI2=0.
      TERMI3=0.
        DO 40 K=1,4
        XC(K) = AX * XI(K) + BX
        YC(K) = AY * XI(K) + BY
        TERMA=TERMA+0.5*(XC(K)*ENX+YC(K)*ENY)*WG(K)*SL
        TERMSX=TERMSX+0.5*YC(K)**2*ENY*WG(K)*SL
        TERMSY=TERMSY+0.5*XC(K)**2*ENX*WG(K)*SL
        TERMIX=TERMIX+1./3.*YC(K)**3*ENY*WG(K)*SL
        TERMIY=TERMIY+1./3.*XC(K)**3*ENX*WG(K)*SL
        TERMIXY=TERMIXY+0.25*XC(K)*YC(K)*(XC(K)*ENX+YC(K)*ENY)
     1
                          *WG(K)*SL
```

```
U1 = (XC(K) * *2 * YC(K) + YC(K) * *3) / 8.
      U1N=(2.*XC(K)*YC(K)*ENX+(XC(K)**2+3.*YC(K)**2)*ENY)/8.
        TERMI1=TERMI1+(UB(I)*U1N-UNB(I)*U1)*WG(K)*SL
      U_2 = (XC(K) * * 3 + XC(K) * YC(K) * * 2) / 8.
      U2N = ((3.*XC(K)**2+YC(K)**2)*ENX+2.*XC(K)*YC(K)*ENY)/8.
        TERMI2=TERMI2+(UB(I)*U2N-UNB(I)*U2)*WG(K)*SL
      U3 = (XC(K) * * 2 + YC(K) * * 2) / 4.
      U3N = (XC(K) * ENX + YC(K) * ENY) / 2.
        TERMI3=TERMI3+(UB(I)*U3N-UNB(I)*U3)*WG(K)*SL
   40
        CONTINUE
      AREA=AREA+TERMA
      SX=SX+TERMSX
      SY=SY+TERMSY
      AIX=AIX+TERMIX
      AIY=AIY+TERMIY
      AIXY=AIXY+TERMIXY
      AI1=AI1+TERMI1
      AI2=AI2+TERMI2
      AI3=AI3+TERMI3
   10 CONTINUE
С
С
   Coordinates of the twist center and Neumann's constant
С
      AA(1,1) = AIX
      AA(1,2) = -AIXY
      AA(1,3) = -SX
      AA(2,1) = -AIXY
      AA (2,2) = AIY
      AA(2,3) = SY
      AA(3,1)=SX
      AA(3,2) = -SY
      AA(3,3) = -AREA
      BB(1) = -AI1
      BB(2) = AI2
      BB(3) = -AI3
      CALL LEQS (AA, BB, 3, KS)
      XTC=BB(1)
      YTC=BB(2)
      CT = BB(3)
      WRITE(*,*)XTC,YTC,CT
С
      RETURN
      END
С
C≖
             С
      SUBROUTINE UINTER (XL,YL,XM,YM,XIN,YIN,UB,UNB,UIN,N,IN,XTC,YTC,CT)
C
С
   This subroutine computes the values of u at the internal points
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), XIN(IN+1), YIN(IN+1)
      DIMENSION UB(N), UNB(N), UIN(IN+1), XM(N), YM(N)
С
      XIN(IN+1)=XTC
      YIN(IN+1)=YTC
С
С
   Compute the values of u at the internal points
с
      DO 10 K=1, IN+1
      UIN(K) = 0.
      DO 20 J=1,N
      JP1=J+1
      CALL DALPHA(XIN(K),YIN(K),XL(J),YL(J),XL(JP1),YL(JP1),RESH)
```

```
CALL RLINTC(XIN(K), YIN(K), XL(J), YL(J), XL(JP1), YL(JP1), RESG)
   20 UIN(K)=UIN(K)+RESH*UB(J)-RESG*UNB(J)
   10 CONTINUE
С
       DO 30 K=1,N
       DX = XL(K+1) - XL(K)
       DY = YL(K+1) - YL(K)
       SL=DSQRT(DX**2+DY**2)
       ENX=DY/SL
       ENY=-DX/SL
       UB(K) = UB(K) - YTC * XM(K) + XTC * YM(K) - CT
       UNB(K) = UNB(K) - YTC*ENX+XTC*ENY
   30 CONTINUE
C
       DO 40 K=1, IN+1
       \texttt{UIN}(\texttt{K}) \neq \texttt{UIN}(\texttt{K}) - \texttt{YTC} \star \texttt{XIN}(\texttt{K}) + \texttt{XTC} \star \texttt{YIN}(\texttt{K}) - \texttt{CT}
   40 CONTINUE
       RETURN
       END
с
С
C=
           ¢
       SUBROUTINE TORSTIF (XL, YL, N, UB, XTC, YTC, D)
С
С
С
   This subroutine computes the torsion constant
С
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION XL(N+1), YL(N+1), UB(N)
      DIMENSION XC(4), YC(4), XI(4), WG(4)
с
       N = Number of boundary elements
C
С
   XL, YL = Coordinates of the extreme points of the boundary elements
С
       D = Torsion constant
С
      WG = Weights of the Gauss integration
С
      XI = Coordinates of the Gauss integration points in the
С
            interval [-1, 1]
С
   XC, YC = Global coordinates of the Gauss integration points
C
      DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
      D=0.
С
      DO 10 I=1,N
      AX = (XL(I+1) - XL(I))/2.
      AY = (YL(I+1) - YL(I)) / 2.
      BX = (XL(I+1) + XL(I)) / 2.
      BY = (YL(I+1) + YL(I))/2.
      SL=DSQRT(AX**2+AY**2)
      ENX=AY/SL
      ENY=-AX/SL
С
      TERM=0.
        DO 40 K=1,4
         XC(K) = AX * XI(K) + BX - XTC
        YC(K) = AY * XI(K) + BY - YTC
        TERM=TERM+WG(K)*((XC(K)*YC(K)**2-YC(K)*UB(I))*ENX
     1
                   + (YC(K) *XC(K) **2+XC(K) *UB(I)) *ENY) *SL
        CONTINUE
   40
      D=D+TERM
   10 CONTINUE
      RETURN
```

END

```
С
С
Casaaaa
        С
      SUBROUTINE TORSTRESS (XL, YL, XM, YM, UB, XTC, YTC, TTZ, SL, N)
С
С
   This subroutine computers the boundary shear stress \tau tz
С
С
  in the tangential direction
С
С
  TTZ = Shear stresses at the boundary nodal points
    SL = Distances between the boundary nodal points
С
С
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N)
      DIMENSION TTZ(N), SL(N), UB(N)
С
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
      DO 10 I=1,N
      AX = (XL(I+1) - XL(I))/2.
      AY = (YL(I+1) - YL(I))/2.
      SL(I) =DSQRT(AX**2+AY**2)
   10 CONTINUE
С
     DO 20 I=1,N
      AX = (XL(I+1) - XL(I))/2.
      AY = (YL(I+1) - YL(I)) / 2
      SSL=DSQRT(AX**2+AY**2)
      ENX=AY/SSL
     ENY = - AX/SSL
С
     IF (I.EQ.1) THEN
      Sl=SL(N)+SL(1)
     S2 = SL(I) + SL(I+1)
     B1=UB(N)
     B2 = UB(1)
     B3 = UB(2)
     ELSE IF (I.EQ.N) THEN
      S1=SL(N-1)+SL(N)
     S2=SL(N)+SL(1)
      B1=UB(N-1)
      B2=UB(N)
     B3=UB(1)
      ELSE
      S1 = SL(I - 1) + SL(I)
      S2 = SL(I) + SL(I+1)
      B1=UB(I-1)
      B2=UB(I)
      B3=UB(1+1)
      ENDIF
     UBT=(S1**2*B3-S2**2*B1+(S2**2-S1**2)*B2)
           /(S1*S2*(S1+S2))
     1
      TTZ(I) = UBT + (XM(I) - XTC) * ENX + (YM(I) - YTC) * ENY
   20 CONTINUE
      RETURN
      END
с
С
C==
       С
      SUBROUTINE OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, D, TTZ, N, IN, XTC, YTC, CT)
С
С
```

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```
C This subroutine prints the results in the outputfile.
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XM(N), YM(N), UB(N), UNB(N)
      DIMENSION XIN(IN+1), YIN(IN+1), UIN(IN+1), TTZ(N)
С
      WRITE(2,100)
  100 FORMAT('',69('*')//1X,'RESULTS'//2X,'BOUNDARY NODES'//
              11X, 'X', 15X, 'Y', 15X, 'U', 14X, 'Un'/)
     1
С
      DO 10 I=1,N
   10 WRITE(2,200) XM(I),YM(I),UB(I),UNB(I)
  200 FORMAT(4(2X,E14.5))
C
      WRITE(2,900)
  900 FORMAT(/'',69('*')//3X,'COORDINATES OF THE TWIST CENTER',
                AND ARBITRARY CONSTANT'/)
             .
     1
     WRITE(2,950)XTC,YTC,CT
  950 FORMAT(3X, 'XTC=', E11.5, 2X, 'YTC=', E11.5, 11X, 'C=', E11.5)
      WRITE(2,500)
С
     WRITE (2,300)
  300 FORMAT(/,2X,'INTERNAL POINTS'//10X,'X',15X,'Y',11X,
             'SOLUTION U'/)
    1
     DO 20 K=1,IN
  20 WRITE(2,400) XIN(K), YIN(K), UIN(K)
  400 FORMAT(3(2X,E14.5))
     WRITE(2,600) D
  600 FORMAT(/,'',69('*')//2X,'TORSION CONSTANT
                                                   D=',E11.5/)
     WRITE(2,700)
  700 FORMAT('',69('*')//2X,'BOUNDARY STRESS Ttz'//
              11X, 'X', 15X, 'Y', 13X, 'Ttz'/)
    1
     DO 30 I=1,N
  30 WRITE(2,800) XM(I),YM(I),TTZ(I)
  800 FORMAT(3(2X,E14.5))
      WRITE(2,500)
  500 FORMAT(/,'',69('*'))
     RETURN
      END
С
С
C____________________________
```

Example 6.1

In this example the program TORSCON is employed to solve the torsion problem for a bar of elliptic cross-section with semi-axes a = 5.0 and b = 3.0. The boundary is discretized into N unequal constant elements as in Example 4.2. The coordinates of the extreme points are computed from the relations

 $x_i = a \cos \theta_i$ and $y_i = b \sin \theta_i$

where

$$\theta_i = -\Delta \theta/2 + (i-1)\Delta \theta$$
, $\Delta \theta = 2\pi/N$ $(i = 1, 2, ..., N)$

The coordinates of the internal points, which are located on concentric ellipses, are computed from the relations

 $\begin{aligned} x_k &= a_i \cos \theta_j \quad \text{and} \quad y_k &= b_i \sin \theta_j \\ k &= (i-1) \operatorname{N2} + j \\ a_i &= i \Delta a , \quad \Delta a &= a/(\operatorname{N1} + 1) \quad (i = 1, 2, \dots, \operatorname{N1}) \\ b_i &= i \Delta b , \quad \Delta b &= b/(\operatorname{N1} + 1) \quad (i = 1, 2, \dots, \operatorname{N1}) \\ \theta_j &= (j-1) \Delta \theta , \quad \Delta \theta &= 2\pi/\operatorname{N2} \quad (j = 1, 2, \dots, \operatorname{N2}) \end{aligned}$

in which N1 denotes the number of internal concentric ellipses and N2 the number of points on each of the ellipses.

The analytical solution yields [1]

$$\phi = \frac{b^2 - a^2}{a^2 + b^2} xy, \qquad I_t = \frac{\pi a^3 b^3}{a^2 + b^2}, \qquad \frac{\tau_{tz}}{G\theta} = \frac{2\sqrt{b^4 x^2} + a^4 y^2}{a^2 + b^2}$$

The data file, which has been produced using program ELLIPSE-3.FOR, and the results of program TORSCON for N = 20, N1 = 1 and N2 = 12 (IN = 12) are presented below.

EXAMPLE 6.1 (DATA)

J.T. Katsikadelis			
Example 6.1			
4.9384417			
4.9384417			
4.4550326			
3 5355330			

4.9384417	4693034
4.9384417	.4693034
4.4550326	1.3619715
3.5355339	2.1213203
2.2699525	2.6730196
.7821723	2.9630650
7821723	2.9630650
-2.2699525	2.6730196
-3.5355339	2.1213203
-4.4550326	1.3619715
-4.9384417	.4693034
-4.9384417	4693034
-4.4550326	-1.3619715
-3.5355339	-2.1213203
-2.2699525	-2.6730196
7821723	-2.9630650
.7821723	-2.9630650
2.2699525	-2.6730196
3.5355339	-2.1213203
4.4550326	-1.3619715
2.5000000	.0000000
2.1650635	.7500000
1.2500000	1.2990381
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.0000000	1.5000000
-1.2500000	1.2990381
-2.1650635	.7500000
-2.5000000	.0000000
-2.1650635	7500000
-1.2500000	-1.2990381
.0000000	-1.5000000
1.2500000	-1.2990381
2.1650635	7500000
.0000000	.0000000

EXAMPLE 6.1 (RESULTS)

BASIC PARAMETERS

NUMBER OF BOUNDARY ELEMENTS= 20 NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED= 13

COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELEMENTS

POINT	XL	YL
1	.49384E+01	46930E+00
2	.49384E+01	.46930E+00
3	.44550E+01	.13620E+01
4	.35355E+01	.21213E+01
5	.22700E+01	.26730E+01
6	.78217E+00	.29631E+01
7	78217E+00	.29631E+01
8	22700E+01	.26730E+01
9	35355E+01	.21213E+01
10	44550E+01	.13620E+01
11	49384E+01	.46930E+00
12	49384E+01	46930E+00
13	44550E+01	13620E+01
14	35355E+01	21213E+01
15	22700E+01	26730E+01
16	78217E+00	29631E+01
17	.78217E+00	29631E+01
18	.22700E+01	26730E+01
19	.35355E+01	21213E+01
20	.44550E+01	13620E+01

BOUNDARY CONDITIONS

NODE	INDEX	PRESCRIBED VALUE
1	1	.00000E+00
2	1	14314E+01
3	1	19716E+01
4	1	17030E+01
5	1	95863E+00
6	1	.00000E+00
7	1	.95863E+00
8	1	.17030E+01
9	1	.19716E+01
10	1	.14314E+01

11	1	.00000E+00
12	1	14314E+01
13	1	19716E+01
14	1	17030E+01
15	1	95863E+00
16	1	.00000E+00
17	1	.95863E+00
18	1	.17030E+01
19	1	.19716E+01
20	0	.00000E+00

The system has been solved regularly

RESULTS

BOUNDARY NODES

х	Y	υ	Un
.49384E+01	.00000E+00	.44409E-15	24373E-16
.46967E+01	.91564E+00	19551E+01	14314E+01
.39953E+01	.17416E+01	31820E+01	19716E+01
.29027E+01	.23972E+01	31957E+01	17030E+01
.15261E+01	.28180E+01	19795E+01	95863E+00
.00000E+00	.29631E+01	11102E-14	.12682E-15
15261E+01	.28180E+01	.19795E+01	.95863E+00
29027E+01	.23972E+01	.31957E+01	.17030E+01
39953E+01	.17416E+01	.31820E+01	.19716E+01
46967E+01	.91564E+00	.19551E+01	.14314E+01
49384E+01	.00000E+00	11102E-14	.24373E-16
46967E+01	91564E+00	19551E+01	14314E+01
39953E+01	17416E+01	31820E+01	19716E+01
29027E+01	23972E+01	31957E+01	17030E+01
15261E+01	28180E+01	19795E+01	95863E+00
.00000E+00	29631E+01	24425E-14	12682E-15
.15261E+01	28180E+01	.19795E+01	.95863E+00
.29027E+01	23972E+01	.31957E+01	.17030E+01
.39953E+01	17416E+01	.31820E+01	.19716E+01
.46967E+01	91564E+00	.19551E+01	.14314E+01
*****	**************	******	***********
COORDINATES OF	THE TWIST CENTER	AND ARBITRARY	CONSTANT
XTC= .12682E-1	5 YTC= .24373E-16	C=19	551E+01
* * * * * * * * * * * * * * * * * * * *	*************		
THERMAL DOTHER			

INTERNAL POINTS

*

*

x	Y	SOLUTION U
.25000E+01	.00000E+00	44409E-15
.21651E+01	.75000E+00	73446E+00
.12500E+01	.12990E+01	~.73517E+00
.00000E+00	.15000E+01	22204E-15
12500E+01	.12990E+01	.73517E+00
21651E+01	.75000E+00	.73446E+00
25000E+01	.00000E+00	88818E-15
21651E+01	75000E+00	73446E+00
12500E+01	12990E+01	73517E+00

.00000E+00	15000E+01	88818E-15	
.12500E+01	12990E+01	.73517E+00	
.21651E+01	75000E+00	.73446E+00	
.00000E+00	.00000E+00	88818E-15	
*****	******	********	***
TORSION CONSTANT	D= .30472E+03		
****	*****	********	**
BOUNDARY STRESS T	tz		
v	v		
A	1	112	
.49384E+01	.00000E+00	.29371E+01	
.46967E+01	.91564E+00	.29825E+01	
.39953E+01	.17416E+01	.32838E+01	
.29027E+01	.23972E+01	.37469E+01	
.15261E+01	.28180E+01	.41137E+01	
.00000E+00	.29631E+01	.42484E+01	
15261E+01	.28180E+01	.41137E+01	
29027E+01	.23972E+01	.37469E+01	
39953E+01	.17416E+01	.32838E+01	
46967E+01	.91564E+00	.29825E+01	
49384E+01	.00000E+00	.29371E+01	
46967E+01	91564E+00	.29825E+01	
39953E+01	17416E+01	.32838E+01	
29027E+01	23972E+01	.37469E+01	
15261E+01	28180E+01	.41137E+01	
.00000E+00	29631E+01	.42484E+01	
.15261E+01	28180E+01	.41137E+01	
.29027E+01	23972E+01	.37469E+01	
.39953E+01	17416E+01	.32838E+01	
.46967E+01	91564E+00	.29825E+01	



Figure 6.3 Contours of the warping function in a bar of elliptic cross-section.

Table 6.1 presents the computed values for several quantities of the elliptic cross-section for various values of N and also helps to draw conclusions about the accuracy of BEM. Moreover, Fig. 6.3 shows the contours of the warping surface $\phi = w/\theta$.

<u> </u>							
Point	Number of boundary elements, N					Exact	
x, y	20	60	100	160	220	300	
	•	Valu	les of ϕ at	internal p	oints		
2.1651, 0.7500	-0.7345	-0.7607	-0.7629	-0.7636	-0.7639	-0.7640	-0.7641
1.2500, 1.2990	-0.7352	-0.7607	-0.7629	0.7637	-0.7639	-0.7640	-0.7641
		Values o	$f \tau_{tz}/G\theta$	at bounda	ary nodes		
5.0000, 0.0000	2.9371	2.6846	2.6608	2.6525	2.6499	2.6486	2.6471
0.0000, 3.0000	4.2484	4.3930	4.4050	4.4091	4.4104	4.4110	4.4118
	Values of I_t						
	304.72	311.08	311.57	311.74	311.79	311.82	311.84

Table 6.1 Computed values of ϕ , $\tau_{tz}/G\theta$ and I_t in a bar of elliptic crosssection for various values of N.

Example 6.2

The program TORSCON is used to solve the torsion problem for a bar with square cross-section of side a = 4.0. The values of I_t and $\max \tau_{tz}/G\theta$ are computed for various values of N and are listed in Table 6.2. The data file is constructed for each case of discretization with program RECT-3.FOR. The warping function $\phi = w/\theta$ is shown in Fig. 6.4 and Fig. 6.5. The exact values have been computed from the analytical expressions derived for the rectangular cross-section $a \times b$ [1]

$$I_{t} = \frac{1}{3}a^{3}b\left[1 - \frac{192}{\pi^{5}}\frac{a}{b}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^{5}}\tanh\left(\frac{n\pi b}{2a}\right)\right]$$
$$\frac{\max\tau_{tz}}{G\theta} = a\left[1 - \frac{8}{\pi^{2}}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^{2}\cosh\left(\frac{n\pi b}{2a}\right)}\right]$$



Figure 6.4 Warping function $\phi = w/\theta$ for a bar of square cross-section.



Figure 6.5 Contours of the warping function in a bar of square cross-section.

Boundary elements, N	I _t	$\max \tau_{\it tz}/ G\theta$
20	35.718	2.6186
60	35.979	2.6938
100	35.988	2.6988
140	35.989	2.7010
220	35.989	2.7010
Exact value	35.990	2.7010

Table 6.2 Computed values of I_t and $\max \tau_{tz}/G\theta$ in a bar of square cross-section for various values of N.

6.2.4 Torsion of anisotropic bars

Let us consider a bar of non-circular cross-section. It consists of an anisotropic material having one plane of material property symmetry, which is perpendicular to the z-axis of the bar (see Fig. 6.1). This means that through any point of the bar passes a plane normal to the z-axis with the property that any two directions symmetric with respect to this plane are directions of material property symmetry. Such a material is termed monoclinic and there are 13 independent elastic constants for this material instead of 21. In this case, the generalized Hooke's law valid for the general anisotropic body may be simplified and be expressed by the following six relations [6]

$$\varepsilon_{x} = \alpha_{11} \sigma_{x} + \alpha_{12} \sigma_{y} + \alpha_{13} \sigma_{z} + \alpha_{16} \tau_{xy}$$

$$\varepsilon_{y} = \alpha_{12} \sigma_{x} + \alpha_{22} \sigma_{y} + \alpha_{23} \sigma_{z} + \alpha_{26} \tau_{xy}$$

$$\varepsilon_{z} = \alpha_{13} \sigma_{x} + \alpha_{23} \sigma_{y} + \alpha_{33} \sigma_{z} + \alpha_{36} \tau_{xy}$$

$$\gamma_{yz} = \alpha_{44} \tau_{yz} + \alpha_{45} \tau_{xz}$$

$$\gamma_{xz} = \alpha_{45} \tau_{yz} + \alpha_{55} \tau_{xz}$$

$$\gamma_{xy} = \alpha_{16} \sigma_{x} + \alpha_{26} \sigma_{y} + \alpha_{36} \sigma_{z} + \alpha_{66} \tau_{xy}$$

$$(6.56)$$

Introducing the strain-displacement equations, Eqs. (6.3), into the above constitutive relations, we find

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \tag{6.57}$$

and

$$\alpha_{55} \tau_{xz} + \alpha_{45} \tau_{yz} = \theta \left(\frac{\partial \phi}{\partial x} - y \right)$$

$$\alpha_{45} \tau_{xz} + \alpha_{44} \tau_{yz} = \theta \left(\frac{\partial \phi}{\partial y} + x \right)$$
(6.58)

Equations (6.58) are solved for τ_{xz} and τ_{yz} yielding

$$\tau_{xz} = \frac{\theta}{|\alpha|} \left[\alpha_{44} \left(\frac{\partial \phi}{\partial x} - y \right) - \alpha_{45} \left(\frac{\partial \phi}{\partial y} + x \right) \right]$$
(6.59a)

$$\tau_{yz} = \frac{\theta}{|\alpha|} \left[-\alpha_{45} \left(\frac{\partial \phi}{\partial x} - y \right) + \alpha_{55} \left(\frac{\partial \phi}{\partial y} + x \right) \right]$$
(6.59b)

where

$$|\alpha| = \det(\alpha) = \begin{vmatrix} \alpha_{55} & \alpha_{45} \\ \alpha_{45} & \alpha_{44} \end{vmatrix} = \alpha_{44} \alpha_{55} - \alpha_{45}^2$$
(6.60)

Introducing Eqs. (6.57) and (6.59) into the equilibrium equations (6.5) and the boundary conditions (6.7), we arrive at following boundary value problem for the warping function ϕ

$$\overline{\alpha}_{44} \frac{\partial^2 \phi}{\partial x^2} - 2 \,\overline{\alpha}_{45} \frac{\partial^2 \phi}{\partial x \partial y} + \overline{\alpha}_{55} \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in } \Omega$$
(6.61)

$$\nabla \phi \cdot \mathbf{m} = y \, m_x - x \, m_y \quad \text{on} \quad \Gamma \tag{6.62}$$

where

$$\mathbf{m} = \left(\overline{\alpha}_{44} n_x - \overline{\alpha}_{45} n_y\right) \mathbf{i} + \left(-\overline{\alpha}_{45} n_x + \overline{\alpha}_{55} n_y\right) \mathbf{j}$$
(6.63)

$$\overline{a}_{44} = \frac{a_{44}}{\sqrt{|a|}}, \quad \overline{a}_{45} = \frac{a_{45}}{\sqrt{|a|}}, \quad \overline{a}_{55} = \frac{a_{55}}{\sqrt{|a|}}$$

m being a vector in the direction of the connormal to the boundary. Eq. (6.61) along with the boundary condition (6.62) allow the determination of the warping function $\phi(x, y)$. Equation (6.61) is of the form of Eq. (3.56) and can be solved using BEM as it was presented in Section 3.6.

The twisting moment at the end cross-sections is

$$M_t = \int_{\Omega} \left(x \tau_{yz} - y \tau_{xz} \right) d\Omega = G I_t \theta$$
(6.64)

where

$$G = \frac{1}{\sqrt{|\alpha|}} \tag{6.65}$$

is a constant having the same dimensions as the shear modulus, and

$$I_{t} = \int_{\Omega} \left[\overline{\alpha}_{55} x^{2} + 2\overline{\alpha}_{45} xy + \overline{\alpha}_{44} y^{2} - \left(\overline{\alpha}_{45} x + \overline{\alpha}_{44} y \right) \frac{\partial \phi}{\partial x} + \left(\overline{\alpha}_{55} x + \overline{\alpha}_{45} y \right) \frac{\partial \phi}{\partial y} \right] d\Omega$$

$$(6.66)$$

The latter may be converted to a line integral on the boundary taking the form

$$I_{t} = \int_{\Gamma} \left\{ \left[\overline{\alpha}_{44} \left(xy^{2} - y\phi \right) + \overline{\alpha}_{45} \left(\frac{1}{2} x^{2} y - x\phi \right) \right] n_{x} + \left[\overline{\alpha}_{55} \left(x^{2} y + x\phi \right) + \overline{\alpha}_{45} \left(\frac{1}{2} xy^{2} + y\phi \right) \right] n_{y} \right\} ds$$

$$(6.67)$$

For orthotropic materials, it is $a_{16} = a_{26} = a_{36} = \alpha_{45} = 0$ and the foregoing equations become

$$\tau_{xz} = \theta G_{xz} \left(\frac{\partial \phi}{\partial x} - y \right)$$

$$\tau_{yz} = \theta G_{yz} \left(\frac{\partial \phi}{\partial y} + x \right)$$
(6.68)

$$\overline{G}_{xz}\frac{\partial^2 \phi}{\partial x^2} + \overline{G}_{yz}\frac{\partial^2 \phi}{\partial y^2} = 0$$
(6.69)

$$\mathbf{m} = \bar{G}_{xz} n_x \,\mathbf{i} + \bar{G}_{yz} n_y \,\mathbf{j} \tag{6.70}$$

$$I_{t} = \int_{\Omega} \left[\overline{G}_{xz} \left(y^{2} - y \frac{\partial \phi}{\partial x} \right) + \overline{G}_{yz} \left(x^{2} + x \frac{\partial \phi}{\partial y} \right) \right] d\Omega$$
$$= \int_{\Gamma} \left[\overline{G}_{xz} \left(xy^{2} - y\phi \right) n_{x} + \overline{G}_{yz} \left(yx^{2} + x\phi \right) n_{y} \right] ds$$
(6.71)

where

$$G_{sz} = \frac{1}{\alpha_{55}}, \quad G_{yz} = \frac{1}{\alpha_{44}}$$
 (6.72)

$$G = \sqrt{G_{xz}G_{yz}} \tag{6.73}$$

$$\overline{G}_{xz} = \overline{a}_{44} = \frac{G_{xz}}{G}, \qquad \overline{G}_{yz} = \overline{a}_{55} = \frac{G_{yz}}{G}$$

$$(6.74)$$

are the non-dimensionalized shear moduli in the xz and yz planes, respectively.

6.3 Deflection of elastic membranes

We consider a flat elastic membrane of uniform thickness h occupying the twodimensional multiply connected domain Ω of the xy-plane bounded by K + 1curves (see Fig. 6.6). The membrane is fixed or elastically supported along its boundary $\Gamma = \bigcup_{i=0}^{i=K} \Gamma_i$ and subjected to a uniform tension S, which is large enough so that it is not appreciably altered when the membrane is deflected by a distributed load f(x, y).



Figure 6.6 Elastic membrane occupying the multiply connected domain Ω .

The equilibrium equation of the deflected membrane can be derived by considering the equilibrium of a deflected element $d\Omega = dxdy$. Here, however, it will be derived using an energy approach, because it allows a better understanding of the linearization of the problem.

The initial flat membrane is deflected to a surface w(x,y) when subjected to a transverse load of density f(x,y). Due to the lateral defection the prestressed membrane is further stretched and additional strains are produced in its middle

surface. The linear deflection theory of membranes is based on the following assumptions:

- (a) The prestress of the membrane is large enough so that the tension S remains unchanged during the deflection.
- (b) The additional strains of the middle surface of the membrane due to its elastic in-plane deformation (u(x,y) and v(x,y)) are negligible compared to those due to the deflection w(x,y) of the middle surface.

The second assumption implies that the strain components are given as

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 \approx \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2$$
(6.75a)

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \approx \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$$
(6.75b)

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \approx \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$
(6.75c)

The strain energy of the deflected membrane is written as

$$U = \frac{h}{2} \int_{\Omega} \left(\sigma_x \varepsilon_x + 2 \tau_{xy} \gamma_{xy} + \sigma_y \varepsilon_y \right) d\Omega$$
(6.76)

or taking into account that $h\sigma_x = h\sigma_y = S$ and $\tau_{xy} = 0$ and using Eqs. (6.75), we arrive at

$$U = \frac{S}{2} \int_{\Omega} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] d\Omega$$
(6.77)

Thus, the total potential energy of the deflected membrane becomes

$$V(u) = \int_{\Omega} \left\{ \frac{S}{2} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] - fw \right\} d\Omega$$
$$+ \int_{\Gamma} \left[\frac{1}{2} k(s) w^2 - R(s) w \right] ds$$
(6.78)

where k(s) is the stiffness modulus of the elastic support and R(s) is the density of the externally applied transverse load along the boundary. The equilibrium equation and the accompanying boundary condition are produced by applying the total potential energy principle, i.e. $\delta V(w) = 0$. Using operations of the calculus of variations and integration by parts, we can easily derive from Eq. (6.78) the following boundary value problem

$$\left.\begin{array}{ccc}S\,\nabla^2 w = -f & \text{in }\Omega\\\beta_1 w + \beta_2 q = \beta_3 & \text{on }\Gamma\end{array}\right\}$$
(6.79)

where $\beta_1 = k(s)$, $\beta_2 = S$, $\beta_3 = R(s)$; $q = \partial w / \partial n$ is the derivative of w along the direction normal to the boundary.

Consequently, the problem of determining the deflection surface of an elastic membrane is reduced to the problem of solving Poisson's equation under Robin's boundary condition. Note that for $k(s) \rightarrow \infty$, the boundary condition of Eq. (6.79) becomes w = 0, namely, the Dirichlet boundary condition.

Example 6.3

Determine the deflection surface of an elastic membrane having the shape of an equilateral triangle with side length a = 5.0 m. The membrane is fixed along its boundary and is subjected to a uniformly distributed load $f = 10 \text{ kN/m}^2$ and a tension S = 1 kN/m. The coordinate axes are taken as shown in Fig. 6.7.



Figure 6.7 Triangular membrane.

The deflection w(x,y) of the membrane is set as

 $w = w_0 + w_1$

where w_0 is the solution of the homogeneous equation and w_1 is a particular solution.

Chapter 6

(i) Particular solution w_1

The particular solution is established by transforming Eq. (6.79) into the complex domain as it was described in Section 3.4.2 and applying Eq. (3.42). Thus, we find

$$w_1 = -\frac{10}{4} \left(x^2 + y^2 \right)$$

(ii) Homogeneous solution w_0

The homogeneous solution will be obtained from the following boundary value problem

$$abla^2 w_0 = 0$$
 in Ω
 $w_0 = rac{10}{4} \left(x^2 + y^2
ight), \qquad (x,y) \in \Gamma$

using the program LABECON.

The analytical solution is [1]



Figure 6.8 Deflection surface of the triangular membrane.

The computed deflections at five internal points along with the corresponding values of the exact solution are given in Table 6.3. Moreover, the computed deflection surface is presented in Fig. 6.8 using contours.

y	Number of boundary elements, N						Exact
	30	60	90	150	210	300	Exact
-0.7217	5.4550	5.4323	5.4284	5.4264	5.4259	5.4256	5.4253
0.0000	6.9743	6.9514	6.9475	6.9455	6.9450	6.9447	6.9444
0.7217	5.8895	5.8663	5.8624	5.8604	5.8599	5.8596	5.8524
1.4434	3.5043	3.4793	4.4753	3.4733	3.4728	3.4725	3.4722
2.1651	1.1159	1.0933	1.0883	1.0862	1.0856	1.0853	1.0851

Table 6.3 Deflections w(0, y) of the triangular membrane on the section x = 0 for various values of N.

6.4 Bending of simply supported plates

The deflection w(x,y) of a thin elastic plate occupying the two-dimensional domain Ω of the xy-plane satisfies the equation [9]

$$\nabla^4 w = \frac{f}{D} \tag{6.80}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$
 is the flexural rigidity of the plate,

- ν is the Poisson's ratio,
- h is the constant thickness of the plate,
- f = f(x, y) is the distributed transverse load, and

$$\nabla^4 = \nabla^2 \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$
 is the biharmonic

operator

The bending and twisting moments are given by the expressions

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$
(6.81a)

$$M_{y} = -D\left(\frac{\partial^{2}w}{\partial y^{2}} + \nu \frac{\partial^{2}w}{\partial x^{2}}\right)$$
(6.81b)

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x \,\partial y}$$
(6.81c)

For a simply supported plate the deflection must satisfy the following boundary conditions on the plate boundary Γ

$$w = 0 \tag{6.82a}$$

$$M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2}\right) = 0$$
(6.82b)

where M_n is the bending moment in the direction n normal to the boundary and t denotes the tangential to the boundary direction.

Note that for a curvilinear boundary it is [10]

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial s^2} + \kappa \frac{\partial w}{\partial n}$$
(6.83)

where $\kappa = \kappa(s)$ is the curvature of the boundary. When the boundary of the simply supported plate consists of rectilinear segments, apparently it is

$$\kappa = 0$$
, $w = 0$, $\frac{\partial w}{\partial s} = 0$, $\frac{\partial^2 w}{\partial s^2} = 0$

In this case, Eq. (6.83) results in

$$\frac{\partial^2 w}{\partial t^2} = 0$$

while Eq. (6.82b) reduces to

$$\frac{\partial^2 w}{\partial n^2} = 0$$

Consequently, on the basis of the above two equations, the deflection should satisfy the following equation on the boundary of the plate

$$\nabla^2 w = \frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{on} \quad \Gamma$$
(6.84)

For points inside the domain Ω , Eqs. (6.81) give

$$M_x + M_y = -D(1+\nu)\nabla^2 w$$

So, by setting

$$M = \frac{M_x + M_y}{1 + \nu} = -D\,\nabla^2 w \tag{6.85}$$

we can write Eq. (6.80) as

$$\nabla^2 \left(-D \, \nabla^2 w \right) = -f$$

which by virtue of Eq. (6.85) may be split into the following two potential equations

$$\nabla^2 M = -f \tag{6.86}$$

$$\nabla^2 w = -\frac{M}{D} \tag{6.87}$$

Equation (6.84) yields M = 0 on the boundary of the plate. Therefore, the solution of Eq. (6.80) for a simply supported plate with a polygonal boundary can be obtained from the following two Dirichlet problems:

$$\left. \begin{array}{c} \nabla^2 M = -f & \text{in } \Omega \\ M = 0 & \text{on } \Gamma \end{array} \right\}$$

$$(6.88)$$

and

$$\nabla^2 w = -\frac{M}{D} \quad \text{in } \Omega \\
 w = 0 \quad \text{on } \Gamma
 \end{cases}$$
(6.89)

The solution of the plate equation by separating it into two potential equations is attributed to Marcus [11]. Its use is limited, because it can treat only simply supported plates with a polygonal boundary. For the extension of the method to plates with a curvilinear boundary under any type of boundary conditions, the reader is referred to the work of Paris and De Leon [12] or Katsikadelis [13]. Nevertheless, the solution of the general plate problem can be obtained by the BEM according to the procedure developed for the biharmonic operator [13].

The solution of Eqs. (6.88) and (6.89) using the BEM requires the evaluation of the domain integrals

$$\int_{\Omega} v f \, d\Omega$$
 and $\int_{\Omega} v M \, d\Omega$

where v is the fundamental solution of the Laplace equation. The function f is known and as a result the first of the above integrals can be evaluated using any of

the methods presented in Section (3.5). The function M, however, is not given by an analytical expression, but by its numerical values at any desired points, which can be used to convert it into a boundary line integral employing the Dual Reciprocity Method (DRM) [14] (see also Section 4.5).

6.5 Heat transfer problems

The heat transfer equation is derived from the energy conservation principle, and its general form is given as [15]

$$-\nabla \cdot \mathbf{q}(\mathbf{x},t) + f(\mathbf{x},t) = \frac{\partial(u\rho)}{\partial t}$$
(6.90)

where $\mathbf{x}(x, y, z) \in \Omega$ and

- t: time
- \mathbf{q} : heat flux
- f: rate of internal heat generation due to heat sources
- ρ : material density
- u: specific internal energy
- Ω : domain occupied by the body

For two-dimensional problems, the heat may flow in any direction of the xy-plane. This flow is described by the *flux vector* \mathbf{q} , whose direction is that of the heat flow and its magnitude expresses the heat passing per unit time through the unit surface normal to the direction of the heat flow.

According to the generalized Fourier's law, the thermal flux density depends linearly on the gradient of the temperature field. Namely, denoting by T = T(x, y, t)the temperature, the flux is expressed as

$$\mathbf{q} = \begin{cases} q_{x} \\ q_{y} \end{cases} = - \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{cases} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{cases} = - \begin{cases} k_{xx} \frac{\partial T}{\partial x} + k_{xy} \frac{\partial T}{\partial y} \\ k_{yx} \frac{\partial T}{\partial x} + k_{yy} \frac{\partial T}{\partial y} \end{cases}$$
(6.91)

or

$$\mathbf{q} = -\mathbf{D} \cdot \nabla T \tag{6.92}$$

The matrix

$$\mathbf{D} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$$
(6.93)

provides information about the heat transfer in any direction and it is referred to as the *conductivity matrix*. It is the *constitutive matrix* for the heat transfer problem. For non-homogeneous materials, the constitutive matrix depends on the position of point $\mathbf{x}(x, y)$, that is $\mathbf{D} = \mathbf{D}(x, y)$. The determinant of the matrix \mathbf{D} does not vanish, i.e.

$$|D| \neq 0 \tag{6.94}$$

In general **D** is not a symmetric matrix. However, for the simplicity of the expressions it is assumed here to be symmetric, thus $k_{xy} = k_{yx}$.

If the material is orthotropic, it will be $k_{xy} = k_{yx} = 0$ and the matrix **D** can be simplified taking the form

$$\mathbf{D} = \begin{bmatrix} k_{xx} & 0\\ 0 & k_{yy} \end{bmatrix}$$
(6.95)

Moreover, for an isotropic material it is $k_{xx} = k_{yy} = k$ and consequently we may write

$$\mathbf{D} = k \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} \tag{6.96}$$

The negative sign in Eq. (6.92) is due to the fact the heat flows from higher to lower temperature regions, while the gradient ∇T is directed towards regions of higher temperature.

It is known from thermodynamics that the internal energy depends linearly on the temperature

$$u = c T(\mathbf{x}, t) \tag{6.97}$$

where c is the specific heat.

Taking into account Eqs. (6.92) and (6.97), Eq. (6.90) becomes

$$\nabla \cdot (\mathbf{D} \cdot \nabla T) + f = \frac{\partial (c T \rho)}{\partial t}$$
(6.98)

If the material properties, i.e. heat conductivity, specific heat and density, do not depend on the temperature, the differential equation (6.98) becomes linear.

In steady-state heat transfer, that is when the thermal equilibrium has been reached, the temperature distribution within the body does not depend anymore on time and Eq. (6.98) simplifies to

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$$\nabla \cdot (\mathbf{D} \cdot \nabla T) + f = 0 \tag{6.99}$$

Further, when the conductivity matrix is constant, that is it does not depend on point \mathbf{x} , the material is thermally homogeneous and Eq. (6.99) takes the form

$$k_{xx}\frac{\partial^2 T}{\partial x^2} + 2k_{xy}\frac{\partial^2 T}{\partial x \partial y} + k_{yy}\frac{\partial^2 T}{\partial y^2} + f = 0$$
(6.100)

To solve the heat transfer problem, Eq. (6.100) needs to be solved subject to appropriate prescribed boundary conditions. These boundary conditions can be of the following three types [16, 17]

$$T = \overline{T} \quad \text{on } \Gamma_{1}$$

$$q_{n} = \overline{q}_{n} \quad \text{on } \Gamma_{2}$$

$$q_{n} = -h_{o}(T - T_{\alpha}) \quad \text{on } \Gamma_{3}$$

$$(\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3} = \Gamma)$$

$$(6.101)$$

where $q_n = \mathbf{q} \cdot \mathbf{n}$ denotes the projection of the heat flux vector on the normal to the boundary, h_o represents the heat transfer coefficient and T_α the ambient temperature outside the body. The first boundary condition is a Dirichlet condition or otherwise known as an *essential* condition. The second is a Neumann condition or otherwise known as a *natural* condition. The third condition, otherwise known as a Robin condition, is a linear relationship between the flux and the temperature on the boundary. It is particularly important in heat transfer applications, as it represents the convection condition.

If the material is orthotropic, $k_{xy} = 0$, Eq. (6.100) becomes

$$k_{xx}\frac{\partial^2 T}{\partial x^2} + k_{yy}\frac{\partial^2 T}{\partial y^2} + f = 0$$
(6.102)

Finally, if the material is isotropic, i.e. $k_{xx} = k_{yy} = k$, Eq. (6.100) and the boundary conditions become

$$k\,\nabla^2 T + f = 0 \tag{6.103}$$

$$T = \overline{T} \quad \text{on} \ \Gamma_1 \tag{6.104a}$$

$$-k\frac{\partial T}{\partial n} = \overline{q}_n \quad \text{on} \quad \Gamma_2$$
(6.104b)

$$-k\frac{\partial T}{\partial n} = -h_o(T - T_\alpha)$$
 on Γ_3 (6.104c)

As it was previously mentioned, the first two boundary conditions are typical Dirichlet and Neumann boundary conditions, respectively, while the third one has the form of Eq. (3.6b), namely

$$\alpha T + \beta \, \frac{\partial T}{\partial n} = \gamma \tag{6.105}$$

where $\alpha = h_o$, $\beta = -k$ and $\gamma = h_o T_\alpha$.

For a numerical solution of the problem with BEM, the third boundary condition, Eq. (6.104c), is treated as follows. Equation (6.105) is applied to all nodal points on part Γ_3 of the boundary, which yields

$$[\alpha]\{T\} + [\beta]\{T_n\} = \{\gamma\}$$
(6.106)

where

$$[\alpha] = h_o[I], \quad [\beta] = -k[I], \quad \{\gamma\} = h_o T_\alpha \begin{cases} 1\\1\\\vdots\\1\\1 \end{cases}$$
(6.107)

in which [I] is the unit matrix of dimensions $N_3 \times N_3$, where N_3 is the number of nodal points on Γ_3 .

It is apparent that all the types of boundary conditions can be written in the form of Eq. (6.106) by specifying appropriately the coefficients α , β and γ , e.g. for the Dirichlet condition it is $\alpha = 1$, $\beta = 0$, $\gamma = \overline{T}$. Thus, Eq. (6.106) along with Eq. (4.7), which for u = T becomes

$$[H]{T} = [G]{T_n}$$
(6.108)

can be combined into a single equation forming a system of 2N linear algebraic equations, which allow the determination of the unknown boundary quantities.

Example 6.4

We consider the circular insulated metal heating duct of Fig. 6.9, in which a liquid of temperature 300° C flows. We have to determine the temperature distribution inside the insulation, when the outside temperature is -15° C. The thermal conductivity k is constant.

The problem can be solved using the BEM for domains with multiple boundaries (see Section 4.7). For the problem at hand, however, we can take advantage of the symmetry of the domain with respect to both the x and y axes and restrict the solution to the *ABCDE* quadrant (Fig. 6.10). The symmetry implies zero flux in

the directions normal to the cuts AB and ED. All the boundary conditions related to the reduced domain are shown in Fig. 6.10.



Figure 6.9 Insulated circular heating duct. Temperature inside $T = 300^{\circ}$ C, outside $T = -15^{\circ}$ C.



Figure 6.10 Quadrant *ABCDE* and boundary conditions ($T_n = \partial T / \partial n$).

Table 6.4	Computed values of the temperature T and its derivatives
	$\partial T/\partial x$ and $\partial T/\partial n$ for various values of N.

Point	Number of boundary elements, N						
	21	41	71	141	211	421	
	<i>T</i>						
1	33.973	34.299	34.278	34.269	34.266	34.263	
	$\partial T/\partial x$						
1	-656.78	-660.77	-661.20	-661.10	-661.07	-661.03	
	$\partial T/\partial n$						
2	3063.9	3102.6	3116.1	3123.7	3125.4	3126.8	



Figure 6.11 Temperature contours in the quadrant ABCDE.

The computed values of the temperature T and its derivative $\partial T/\partial x$ at the internal point 1 (0.175, 0.175), and of the normal derivative $\partial T/\partial n$ at the boundary point 2 ($0.1\sqrt{2}/2, 0.1\sqrt{2}/2$) are given in Table 6.4 for various values of the number of constant boundary elements (nodal points) N. Finally, the contour plots of the temperature distribution in the insulation are depicted in Fig. 6.11.

6.6 Fluid flow problems

An ideal or perfect fluid is one that has zero viscosity and is incompressible. A good approximation to the solution of the inviscid (nonviscous) fluid flow problem is achieved by satisfying only the continuity equation, which for a two dimensional flow is written as [18]

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} + f = 0 \tag{6.109}$$

where ρ is the density of the fluid, $\mathbf{v}(v_x, v_y)$ the velocity of the fluid at point (x, y) and f = f(x, y) the distribution of a possible internal source. When the density is constant (incompressible fluid) Eq. (6.109) becomes

$$\nabla \cdot \mathbf{v} + f/\rho = 0 \tag{6.110}$$



Figure 6.12 Two-dimensional fluid flow.

Consider the two-dimensional fluid flow of Fig. 6.12. If the flow is irrotational, the vorticity is zero and the velocity satisfies the equation

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \mathbf{k} = 0 \tag{6.111}$$

or, consequently

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

Therefore, a potential function ϕ exists, which produces the velocity field as

$$\mathbf{v} = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j}$$
(6.112)

and the velocity components are

$$v_x = \frac{\partial \phi}{\partial x}$$
 and $v_y = \frac{\partial \phi}{\partial y}$ (6.113)

Obviously, Eq. (6.111) is satisfied identically and Eq. (6.110) is written as

$$\nabla \cdot \nabla \phi + f/\rho = 0 \tag{6.114}$$

or

$$\nabla^2 \phi + f/\rho = 0 \tag{6.115}$$

The solution of the differential equation (6.115) under boundary conditions which will be presented next, allows the determination of the potential ϕ . Thereafter, the velocity components are obtained from Eqs. (6.113).

The boundary conditions on the velocity potential may be derived from physical considerations. For this purpose, we consider the irrotational and inviscid flow inside a part of a pipe, as shown in Fig. 6.12.

As there is no penetration through the rigid walls AB and DC of the pipe, the normal component v_n of the velocity is zero,

$$v_n = \frac{\partial \phi}{\partial n} = 0 \tag{6.116}$$

Along cross-section AD (inlet) the distribution of v_n may be given, namely

$$v_n = \frac{\partial \phi}{\partial n} = \overline{v}_n \tag{6.117}$$

while at the cross-section BC (outlet) the conditions are:

(i) The velocity component v_n may be prescribed, but its distribution should satisfy along with Eqs. (6.116) and (6.117) the mass conservation principle in the given domain, namely

$$\int_{\Gamma} v_n ds = -\int_{AD} v_n ds + \int_{BC} v_n ds = -\int_{\Omega} f d\Omega$$
(6.118)

(ii) The condition of tranquility of flow (fully developed laminar flow) may be imposed, provided that the cross-section BC is placed sufficiently far from regions of intense variation in the velocity field. This condition is expressed mathematically as

$$\frac{\partial v_n}{\partial n} = \frac{\partial^2 \phi}{\partial n^2} = 0 \tag{6.119}$$

The latter condition was not included in those of Chapter 3 and thus, it requires special treatment.

If section BC is placed away from regions of intense variation of the velocity, as for the laminar flow of case (ii), and its shape is such that the velocity is everywhere normal to it (e.g. straight line), the tangential component of the velocity vanishes. Therefore, it is

$$v_t = \frac{\partial \phi}{\partial s} = 0$$

which implies

$$\phi = C \tag{6.120}$$

where C is an arbitrary constant. This condition permits the establishment of the velocity potential ϕ to the approximation of an arbitrary constant. This, however, does not influence the velocity field, which is determined from Eqs. (6.113).

Apart from the boundary condition (6.119), program LABECON can be utilized to solve fluid flow problems, if it is supplemented by a subroutine that evaluates the derivatives of the potential at internal points according to Eqs. (4.15) and (4.16). The modified LABECON is given the name FLUIDCON. The listing of the main program as well as the subroutines DERIV, which computes the derivatives at the internal points, and OUTPUT, which prints the results, are given below.

```
С
     PROGRAM FLUIDCON
С
С
C This program solves the two dimensional Laplace equation
C for two-dimensional irrotational inviscid flow of (FLUID)s
С
 using the boundary element method with (CON) stant
C boundary elements
С
C
     IMPLICIT REAL*8 (A-H,O-Z)
     CHARACTER*15 INPUTFILE, OUTPUTFILE
С
С
 Set the maximum dimensions
С
     PARAMETER (N=240)
     PARAMETER (IN=50)
C
```

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```
C N\approx Number of boundary elements equal to number of boundary
С
      nodes
C IN= Number of internal points where the function u is calculated
С
      DIMENSION INDEX(N)
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), G(N,N), H(N,N)
      DIMENSION UB(N), A(N, N), UNB(N), XIN(IN), YIN(IN), UIN(IN)
      DIMENSION UXIN(IN), UYIN(IN)
С
С
  Read the names and open the input and output files
С
      WRITE (*,'(A)')' Name of the INPUTFILE (max.15 characters)'
      READ (*, '(A)') INPUTFILE
      WRITE (*, '(A)')' Name of the OUTPUTFILE (max.15 characters)'
      READ (*, '(A)') OUTPUTFILE
      OPEN (UNIT=1, FILE=INPUTFILE)
      OPEN (UNIT=2, FILE=OUTPUTFILE)
С
   Read date from INPUTFILE
С
С
      CALL INPUT (XL, YL, XIN, YIN, INDEX, UB, N, IN)
С
   Compute the G matrix
С
С
      CALL GMATR (XL, YL, XM, YM, G, N)
С
  Compute the H matrix
С
С
      CALL HMATR (XL, YL, XM, YM, H, N)
С
   Form the system of equations AX=B
С
С
      CALL ABMATR (G, H, A, UNB, UB, INDEX, N)
С
С
   Solve the system of equations
C
      CALL SOLVEQ(A, UNB, N, LSING)
С
  Form the vectors U and UN of all the boundary values
С
С
      CALL REORDER (UB, UNB, INDEX, N)
С
   Compute the values of U at the internal points
C
С
      CALL UINTER (XL, YL, XIN, YIN, UB, UNB, UIN, N, IN)
С
С
   Compute the values of Ux and Uy at the internal points
С
      CALL DERIV(XL,YL,XIN,YIN,UB,UNB,UXIN,UYIN,N,IN)
С
С
   Print the results in the OUTPUTFILE
С
      CALL OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, UXIN, UYIN, N, IN)
С
С
   Close input and output files
С
      CLOSE(1)
      CLOSE(2)
      STOP
      END
С
Casas
          С
      SUBROUTINE DERIV(XL,YL,XIN,YIN,UB,UNB,UXIN,UYIN,N,IN)
C
```

```
С
С
  This subroutine computes the values of the derivatives
С
   Ux and Uy at the internal points
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), UB(N), UNB(N)
      DIMENSION XIN(IN), YIN(IN), UXIN(IN), UYIN(IN)
      DIMENSION XC(4), YC(4), XI(4), WG(4)
С
   N
             =Number of boundary elements
С
   XL,YL
             =Coordinates of the extreme points of the
С
               boundary elements
С
   XIN,YIN
            =Coordinates of the internal points
             =Boundary values of U and Un
С
  UB, UNB
  UXIN, UYIN=Derivatives Ux and Uy at the internal points
С
С
  WG
             =Weights of the Gauss integration
¢
  XI
             =Coordinates of the Gauss integration points
С
               in the interval [-1,1]
С
   XC,YC
             =Global coordinates of the Gauss integration points
C
      DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      PI=ACOS(-1.)
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
      PI=ACOS(-1.)
      DO 20 I=1, IN
      UXIN(I) = 0.
      UYIN(I) = 0.
      DO 10 J=1,N
      AX = (XL(J+1) - XL(J)) / 2.
      AY = (YL(J+1) - YL(J))/2.
      BX = (XL(J+1) + XL(J)) / 2.
      BY = (YL(J+1) + YL(J)) / 2.
      SL=SQRT(AX**2+AY**2)
      ANGLE=ATAN2(AY,AX)-PI/2.
      ENX=COS (ANGLE)
      ENY=SIN (ANGLE)
      GX = 0.
      GY = 0.
      HX = 0.
      HY=0.
        DO 40 K=1,4
        XC(K) = AX * XI(K) + BX
        YC(K) = AY*XI(K) + BY
        DX=XC(K)-XIN(I)
        DY=YC(K)-YIN(I)
        R=SQRT(DX**2+DY**2)
        RX = -DX/R
        RY = -DY/R
        RN = -(RX * ENX + RY * ENY)
        RT=-(-RX*ENY+RY*ENX)
        UX=RX/(2.*PI*R)
        UY=RY/(2.*PI*R)
        UNX=-(RX*RN-RY*RT)/(2.*PI*R**2)
        UNY=-(RX*RT+RY*RN)/(2.*PI*R**2)
        GX=GX+UX*WG(K)*SL
        GY=GY+UY*WG(K)*SL
        HX=HX+UNX*WG(K)*SL
        HY=HY+UNY*WG(K)*SL
   40
        CONTINUE
      UXIN(I)=UXIN(I)+HX*UB(J)-GX*UNB(J)
      UYIN(I) = UYIN(I) + HY*UB(J) - GY*UNB(J)
   10 CONTINUE
   20 CONTINUE
      RETURN
```

```
END
С
CHERER
                       С
      SUBROUTINE OUTPUT (XM, YM, UB, UNB, XIN, YIN, UIN, UXIN, UYIN, N, IN)
с
C
С
  This subroutine prints the results in the outputfile
С
     IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XM(N), YM(N), UB(N), UNB(N)
     DIMENSION XIN(IN), YIN(IN), UIN(IN), UXIN(IN), UYIN(IN)
C
     WRITE(2,100)
  100 FORMAT('',79('*')//1X,'RESULTS'//2X,'BOUNDARY NODES'//
             11X, 'X', 15X, 'Y', 15X, 'U', 15X, 'Un'/)
    1
С
     DO 10 I=1,N
  10 WRITE(2,200) XM(I),YM(I),UB(I),UNB(I)
  200 FORMAT(4(2X,E14.5))
С
     WRITE (2,300)
  300 FORMAT(//,2X,'INTERNAL POINTS'//10X,'X',15X,'Y',14X,
    1 'U',14X,'UX',16X,'UY'/)
     DO 20 K=1, IN
C
  20 WRITE(2,400) XIN(K), YIN(K), UIN(K), UXIN(K), UYIN(K)
  400 FORMAT(5(2X,E14.5))
     WRITE(2,500)
  500 FORMAT(' ',79('*'))
     RETURN
     END
С
**************
```

Example 6.5

Consider the steady laminar flow of an incompressible and nonviscous fluid through the tube of Fig. 6.13. Determine the velocity potential and the velocity field in the tube. The shape of the curvilinear segments are determined by the equations:

```
Segment BC: y = 2x^3 - 3x^2 - 1
Segment B'C': y = -2x^3 + 3x^2 + 1
```

The assumed boundary conditions are:

at the inlet cross-section AA': $\frac{\partial \phi}{\partial n} = v_n = -1$

at the outlet cross-section DD': $\phi = 0$

at the rigid walls ABCD and A'B'C'D': $\frac{\partial \phi}{\partial n} = v_n = 0$



Figure 6.13 The tube segment of Example 6.5.

The problem is solved using the program FLUIDCON. The boundary is divided into constant boundary elements in the following way: N1 elements on each of the segments OA and AB; 2*N1 elements on each of the segments BC, CD and DO'. The same discretization is employed for the corresponding symmetric segments above the x-axis. Thus, the total number of elements is N=16*N1.

Figure 6.14 depicts the distribution of the normal velocity component v_n at the outlet cross-section DD' for various values of N. We notice that the flow becomes sufficiently smooth at this cross-section, since it reaches the anticipated mean value $v_n = 0.5$, which is half of the velocity at the inlet cross-section AA'. Moreover, the distributions of the velocity components v_x and v_y at characteristic cross-sections of the tube are shown in Figs. 6.15 and 6.16, while the contour plots of the potential are shown in Fig. 6.17

6.7 Conclusions

In this chapter, we studied a variety of important engineering problems, which are formulated as boundary value problems for the Laplace or Poisson equation. Other problems described by these equations are, for example, the fluid flow through porous media (Darcy's law), diffusion of ions (Fick's law), electric potential in a body (Ohm's law), etc. For all these problems, the flux \mathbf{q} is expressed by a law quite analogous to that of Fourier (see Section 6.5). Of course, the field function u and the constitutive matrix \mathbf{D} have a different physical meaning in each problem. Table 6.5 presents some of the problems mentioned above by giving the physical meaning of the involved quantities.



Figure 6.14 Distribution of the normal velocity component v_n at the outlet cross-section for various values of N.



Figure 6.15 Distribution of the velocity component v_x at characteristic cross-sections of the tube.



Figure 6.16 Distribution of the velocity component v_y at characteristic cross-sections of the tube.



Figure 6.17 Contour plots of the velocity potential ϕ .

A general conclusion that can be drawn from the numerical examples presented in this chapter, is that the BEM with constant elements provides a very good numerical solution for problems described by the Laplace or the Poisson equation. The obtained accuracy is very good. (see Examples 6.1, 6.2, 6.3). Moreover, the preparation of the data is very simple, since the discretization is limited only to the boundary.

Differential equation	Physical problem	Physical quantities	Constitutive relations
$\nabla \cdot (\nabla \phi) = 0$	Saint-Venant torsion of elastic bars	$\phi = \phi(x, y) =$ warping function	Hooke $ \begin{cases} \tau_{sz} \\ \tau_{yz} \end{cases} = [D] \begin{cases} \gamma_{sz} \\ \gamma_{yz} \end{cases} $ $ \mathbf{D} = \mathbf{a}^{-1} $
$\nabla \cdot (S\nabla u) + f = 0$	Small deflections of membranes	u = u(x, y) = deflection surface f = f(x, y) = transverse load S = constant tension force per unit length	$\mathbf{D} = S \mathbf{I}$ $\mathbf{I} = \text{unit matrix}$
$\nabla \cdot (\mathbf{D} \cdot \nabla T) + f = 0$	Heat flow	T = T(x, y) = temperature D = constitutive matrix for thermal conductivity f = f(x, y) = internal heat source density	Fourier $\mathbf{q} = -\mathbf{D} \cdot \nabla T$ $\mathbf{q} = \text{heat flux}$ vector
$\nabla \cdot (\nabla \phi) + f = 0$	Irrotational, incompressible and inviscid fluid flow	$\phi = \phi(x, y) =$ potential function of the velocity field f = internal heat source density	$\mathbf{v} = \nabla \phi$ $\mathbf{v} = \text{velocity}$ There is no constitutive law
$\nabla \cdot (\mathbf{D} \cdot \nabla \phi) + f = 0$	Fluid flow through porous media	f = internal heat source density ϕ = piezometric head \mathbf{D} = constitutive matrix for permeability coefficients	Darcy $\mathbf{q} = -\mathbf{D} \cdot \nabla \phi$ $\mathbf{q} = \text{volume flux}$ vector
$\nabla \cdot (\mathbf{D} \cdot \nabla V) + f = 0$	Electric potential in bodies	 V = potential D = constitutive matrix for electric conductivity f = internal source density of electric charge 	Ohm $\mathbf{q} = -\mathbf{D} \cdot \nabla V$ $\mathbf{q} = \text{electric charge}$ flux vector

Table 6.5Examples of physical problems described by the Laplace or
Poisson equation.

6.8 References

There is a voluminous literature on the topics of this chapter. Therefore, we can only make reference to some well-known and commonly used books. The Saint-Venant torsion problem of non-circular prismatic bars is clearly treated in the books of Timoshenko and Goodier [1] and Muskhelishvili [2]. The reader, however, may supplement his knowledge on this subject by studying the books of Kollbrunner and Basler [3], Friemann [4] and Novozhilov [7]. For heat flow problems the interested reader is referred to the books of Carslaw and Jaeger [16] and Uzisik [17]. Finally, we mention the book of Hirsch for fluid flow problems [18].

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Problems

6.1. Compute the non-dimensionalized torsional constant I_t/h^4 and distribution of stress $\tau_{tz}/G\theta h$ on the boundary of a bar having the cross-sections shown below ($\rho = h/4$). Also, compare the obtained results to those of analytical or approximate solutions [19].



Figure P6.1

- **6.2.** Determine the deflection surface of a rectangular membrane $(0 \le x \le a, 0 \le y \le b)$, which is subjected to hydrostatic pressure $f = q_o x/a$. It is given a = 3.00 m, b = 2.00 m, $q_o = 1 \text{ kN/m}^2$, S = 10 kN/m.
- **6.3.** Solve the problem of Example 6.4 using the whole domain of Fig. 6.9 (two boundaries) instead of using just the upper-right quadrant of Fig. 6.10 (one boundary).

6.4. Determine the temperature distribution in the cross-section of the T-beam insulated as shown in Fig. P6.4.



Figure P6.4

6.5. Solve the torsion problem for the composite bar of Fig. P6.5, if $G_2 = 2G_1$, where G_1 and G_2 are the shear moduli in the subregions Ω_1 and Ω_2 , respectively. Compare the results with those of a homogeneous bar having the same dimensions and $G = 1.5G_1$.



Figure P6.5

- **6.6.** Solve the torsion problem for a bar consisting of an orthotropic material with $G_{yz} = 2G_{xz}$ and having square cross-section of side a = 0.20.
- 6.7. Determine the velocity distribution at the outlet cross-section of the tube in Fig. P6.7, if the constant velocity at the inlet cross-section is $v_n = -1$.



Figure P6.7

Chapter 7

The BEM for Two-Dimensional Elastostatic Problems

7.1 Introduction

This chapter presents the BEM for the solution of linear elastostatic problems in two dimensions. The development of the BEM for the plane elastostatic problems is analogous to that for the plane potential problems discussed in previous chapters. There is, however, an essential difference. Here the problem is formulated in terms of two basic unknowns which are the two displacement components. Therefore, the resulting boundary integral equations are two and coupled, in contrast to the potential problems for which only one integral equation has to be solved. The consequence of this is that the establishment of the fundamental solution as well as its form is much more complicated. Both problems of plane elasticity, namely the plane strain and the plane stress, will be studied in this chapter. Applications will be presented to demonstrate the efficiency and the usefulness of the BEM for solving engineering problems.

7.2 Equations of plane elasticity

7.2.1 Plane strain

Plane strain in linear elasticity is considered the case for which:

- (a) One of the three displacement components, say w along the z-axis, is constant.
- (b) The other two displacements, u and v along the x- and y-axes, respectively, are functions only of the two variables, x and y.

This state of deformation appears in infinitely long (practically very long) prismatic or cylindrical bodies, whose axis coincides with the z-axis, and the loading
is normal to this axis and independent of the z variable (Fig. 7.1). Another example of plane strain is the deformation that occurs in a plane through the axis of a circular cylinder (diametric plane) when the loading is axisymmetric and does not vary in the axial direction.



Figure 7.1 Cross-section of a long dam under plane strain.

7.2.1.1 Kinematic relations

The previously mentioned conditions are stated mathematically as

$$w = C$$
, $u = u(x, y)$, $v = v(x, y)$ (7.1)

where C is an arbitrary constant.

Thus, the components of the strain tensor are [1, 2]

$$\varepsilon_{x} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{y} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{z} = 0, \quad \gamma_{xz} = 0, \quad \gamma_{yz} = 0$$

$$(7.2)$$

7.2.1.2 Constitutive relations

Assuming linearly elastic and isotropic material, the constitutive relations for plane strain are

$$\sigma_{x} = \lambda(\varepsilon_{x} + \varepsilon_{y}) + 2\mu\varepsilon_{x}$$

$$\sigma_{y} = \lambda(\varepsilon_{x} + \varepsilon_{y}) + 2\mu\varepsilon_{y}$$

$$\sigma_{z} = \lambda(\varepsilon_{x} + \varepsilon_{y})$$

$$\tau_{xy} = \mu\gamma_{xy}$$

$$\tau_{xz} = 0$$

$$\tau_{yz} = 0$$

$$(7.3)$$

where λ and μ are the Lamé constants, which are related to the elastic constants E, G and ν through the following expressions

$$\mu = G = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$
(7.4)

We readily conclude by combining Eqs. (7.2) and (7.3) that the non-vanishing components of the stress tensor are functions only of the variables x and y.

Equations (7.3) can be solved for the strain components, yielding

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)]$$
(7.5a)

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_x + \sigma_z)]$$
(7.5b)

$$\sigma_z - \nu(\sigma_x + \sigma_y) = 0 \tag{7.5c}$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \tag{7.5d}$$

$$\gamma_{sz} = 0 \tag{7.5e}$$

$$\gamma_{yz} = 0 \tag{7.5f}$$

Equation (7.5c) gives

 $\sigma_z = \nu(\sigma_x + \sigma_y)$

Substituting this expression for σ_z into Eqs. (7.5a) and (7.5b), we obtain the following two equations for the normal strains

$$\varepsilon_x = \frac{1 - \nu^2}{E} \left(\sigma_x - \frac{\nu}{1 - \nu} \sigma_y \right)$$
$$\varepsilon_y = \frac{1 - \nu^2}{E} \left(\sigma_y - \frac{\nu}{1 - \nu} \sigma_x \right)$$

Setting

$$\overline{\nu} = \frac{\nu}{1-\nu}$$
 and $\overline{E} = \frac{E}{1-\nu^2}$ (7.6)

we find that

$$\mu = G = \frac{E}{2(1+\nu)} = \frac{\overline{E}}{2(1+\overline{\nu})} \quad \text{and} \quad \lambda = \frac{\overline{\nu}\overline{E}}{1-\overline{\nu}^2}$$
(7.7)

and the strain components may then be written as

$$\varepsilon_x = \frac{1}{\overline{E}} (\sigma_x - \overline{\nu} \sigma_y) \tag{7.8a}$$

$$\varepsilon_y = \frac{1}{\overline{E}} (\sigma_y - \overline{\nu} \sigma_x)$$
(7.8b)

$$\gamma_{xy} = \frac{2(1+\overline{\nu})}{\overline{E}} \tau_{xy} \tag{7.8c}$$

The elastic constants \vec{E} and $\vec{\nu}$ are referred to as *effective elastic constants*. As we will see later, the effective elastic constants allow to use equations of the same form for both plane strain and plane stress problems.

Equations (7.8) are combined in a matrix equation as

$$\{\varepsilon\} = [S]\{\sigma\} \tag{7.9}$$

where $\{\varepsilon\}$ and $\{\sigma\}$ are referred to as the strain and stress vectors, respectively, and are defined as

$$\{\varepsilon\} = \begin{cases} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{cases}$$
(7.10)

and

$$\{\sigma\} = \begin{cases} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{cases}$$
(7.11)

The matrix [S] is known as the *compliance or flexibility matrix* and has the form

$$[S] = \frac{1}{\overline{E}} \begin{bmatrix} 1 & -\overline{\nu} & 0\\ -\overline{\nu} & 1 & 0\\ 0 & 0 & 2(1+\overline{\nu}) \end{bmatrix}$$
(7.12)

Solving Eq. (7.9) for the stress vector $\{\sigma\}$, we obtain

$$\{\sigma\} = [C]\{\varepsilon\} \tag{7.13}$$

where

$$[C] = [S]^{-1} = \frac{\overline{E}}{1 - \overline{\nu}^2} \begin{bmatrix} 1 & \overline{\nu} & 0 \\ \overline{\nu} & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \overline{\nu}) \end{bmatrix}$$
(7.14)

The matrix [C] is known as the *stiffness matrix*. The component form of Eq. (7.13) is

$$\sigma_{x} = \frac{\overline{E}}{1 - \overline{\nu}^{2}} (\varepsilon_{x} + \overline{\nu}\varepsilon_{y})$$

$$\sigma_{y} = \frac{\overline{E}}{1 - \overline{\nu}^{2}} (\varepsilon_{y} + \overline{\nu}\varepsilon_{x})$$

$$\tau_{xy} = \frac{\overline{E}}{2(1 + \overline{\nu})} \gamma_{xy}$$

$$(7.15)$$

7.2.1.3 Equilibrium equations

The equilibrium equations for a three-dimensional body are [2]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0$$
(7.16a)

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0$$
(7.16b)

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0$$
(7.16c)

where b_x , b_y and b_z are the components of the body force per unit volume. For the plane problem, it is

$$b_x = b_x(x,y), \quad b_y = b_y(x,y), \quad b_z = 0$$
(7.17)

Taking into account that the stress components are independent of z and using Eqs. (7.3), we readily conclude that the third of Eqs. (7.16) is identically satisfied, while the first two become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$
(7.18)

Substituting Eqs. (7.15) into Eqs. (7.18) and using Eqs. (7.2), we derive the equilibrium equations in terms of the displacement components as

$$\nabla^{2}u + \frac{1 + \overline{\nu}}{1 - \overline{\nu}} \left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}v}{\partial x \partial y} \right) + \frac{1}{G} b_{x} = 0$$

$$\nabla^{2}v + \frac{1 + \overline{\nu}}{1 - \overline{\nu}} \left(\frac{\partial^{2}u}{\partial x \partial y} + \frac{\partial^{2}v}{\partial y^{2}} \right) + \frac{1}{G} b_{y} = 0$$
(7.19)

or substituting $\bar{\nu}$ from Eq. (7.6), we obtain the governing equations for the plane strain problem in the form

$$\nabla^{2}u + \frac{1}{1 - 2\nu} \left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}v}{\partial x\partial y} \right) + \frac{1}{G} b_{x} = 0$$

$$\nabla^{2}v + \frac{1}{1 - 2\nu} \left(\frac{\partial^{2}u}{\partial x\partial y} + \frac{\partial^{2}v}{\partial y^{2}} \right) + \frac{1}{G} b_{y} = 0$$
(7.20)

Equations (7.20) are known as the *Navier equations of equilibrium* for the plane elastostatic problem of a body occupying the two-dimensional domain Ω .

7.2.1.4 Boundary conditions

The solution of Eqs. (7.19) must satisfy prescribed boundary conditions on the boundary Γ of the body, which are based either on the displacements u and v, or on the boundary tractions t_x and t_y . The boundary conditions can be classified into the following four types:

(i)
$$u = \overline{u}, \quad v = \overline{v} \quad \text{on } \Gamma_1$$

(ii) $u = \overline{u}, \quad t_y = \overline{t_y} \quad \text{on } \Gamma_2$
(iii) $t_x = \overline{t_x}, \quad v = \overline{v} \quad \text{on } \Gamma_3$
(iv) $t_x = \overline{t_x}, \quad t_y = \overline{t_y} \quad \text{on } \Gamma_4$
(7.21)

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. The prescribed quantities are designated by an over-bar. Of course, any of the boundary parts Γ_1 , Γ_2 , Γ_3 , Γ_4 may be identical to the whole boundary Γ , that is the boundary conditions may be only of one type. The boundary conditions are mixed, if different boundary conditions are prescribed over two or more parts of the boundary. Attention should be paid when $\Gamma_4 = \Gamma$. In this case, the boundary tractions \overline{t}_x and \overline{t}_y can not be prescribed arbitrarily, but they must ensure the overall equilibrium of the body, namely

$$\int_{\Omega} b_x \, d\Omega + \int_{\Gamma} t_x \, ds = 0$$
$$\int_{\Omega} b_y \, d\Omega + \int_{\Gamma} t_y \, ds = 0$$
$$\int_{\Omega} (x \, b_y - y \, b_x) \, d\Omega + \int_{\Gamma} (x \, t_y - y \, t_x) \, ds = 0$$

For this type of boundary conditions, the solution of the Navier equations is not uniquely determined as it contains an arbitrary rigid body motion. An example of boundary conditions for an infinitely long body of rectangular cross-section is shown in Fig. 7.2.



Figure 7.2 Support and loading conditions on the boundary of a rectangular domain.

The relations between the boundary tractions t_x , t_y and the stress components σ_x , σ_y and τ_{xy} may be derived from the equilibrium of an infinitesimal body element on the boundary (*Cauchy tetrahedron* [2]). In two dimensions these relations are

$$\left. \begin{array}{l} t_x = \sigma_x \, n_x + \tau_{xy} \, n_y \\ t_y = \tau_{xy} \, n_x + \sigma_y \, n_y \end{array} \right\}$$

$$(7.22)$$

where n_x and n_y are the direction cosines of the unit vector normal to the boundary. If the stress components in Eqs. (7.22) are replaced by the expressions of Eqs. (7.3) and then the kinematic relations (7.2) are employed, we arrive at the following displacement-based expressions for the boundary tractions

$$t_{x} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_{x} + \mu \left(\frac{\partial u}{\partial x} n_{x} + \frac{\partial v}{\partial x} n_{y} \right) + \mu \frac{\partial u}{\partial n}$$

$$t_{y} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_{y} + \mu \left(\frac{\partial u}{\partial y} n_{x} + \frac{\partial v}{\partial y} n_{y} \right) + \mu \frac{\partial v}{\partial n}$$

$$(7.23)$$

7.2.1.5 Initial stresses and strains

In many problems initial state of stress or strain may be present, which are due to temperature variations or other causes. For instance, we consider an initial state of strain, whose components are denoted by

$$\{\varepsilon_0\} = \begin{cases} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{cases}$$
(7.24)

Denoting by $\{\varepsilon_t\}$ the total strain, the elastic strain $\{\varepsilon_e\}$ is obtained by subtracting the initial strain from the total one, i.e.

$$\{\varepsilon_e\} = \{\varepsilon_t\} - \{\varepsilon_0\} \tag{7.25}$$

By means of Eq. (7.13), we obtain

$$\{\sigma_c\} = [C]\{\varepsilon_c\} = [C](\{\varepsilon_t\} - \{\varepsilon_0\})$$

or

$$\{\sigma_c\} = [C]\{\varepsilon_t\} - [C]\{\varepsilon_0\} = \{\sigma_t\} - \{\sigma_0\}$$

$$(7.26)$$

where the stresses $\{\sigma_0\} = [C]\{\varepsilon_0\}$ are the initial stresses.

When the initial strain is due to a temperature variation, it is

$$\{\varepsilon_0\} = \alpha \,\Delta T \, \begin{cases} 1\\1\\0 \end{cases} \tag{7.27}$$

where α is the coefficient of thermal dilatation and ΔT the temperature change. The resulting initial stresses are

$$\{\sigma_0\} = [C]\{\varepsilon_0\} = \frac{\overline{E} \ \alpha \ \Delta T}{1 - \overline{\nu}} \begin{cases} 1\\ 1\\ 0 \end{cases}$$
(7.28)

7.2.2 Plane stress

The theory of plane elasticity is applied to another problem of great practical significance, to that of the analysis of thin plates subjected to in-plane loading. Such a state of stress occurs in *shear walls*. We consider a thin elastic body, whose thickness h is very small compared to the other two dimensions (Fig. 7.3).



Figure 7.3 Thin elastic body.

The loading is due to the body forces b_x , b_y and the boundary tractions t_x , t_y . The tractions are usually assumed to be symmetrically distributed with respect to the mid-plane of the body, although more often the variation along the thickness h is considered to be constant. In this case the resulting state of stress is not independent of z, but, if the thickness h is very small, it is quite accurate to assume [2] that

$$\sigma_z=0\,,\quad \tau_{xz}=0\,,\quad \tau_{yz}=0$$

along the thickness h, whereas the remaining stress components do not depend on the variable z, i.e.

$$\sigma_x=\sigma_x(x,y), \quad \sigma_y=\sigma_y(x,y), \quad au_{xy}= au_{xy}(x,y)$$

Thus, the constitutive equations become

$$\varepsilon_{x} = \frac{1}{E} (\sigma_{x} - \nu \sigma_{y})$$

$$\varepsilon_{y} = \frac{1}{E} (\sigma_{y} - \nu \sigma_{x})$$

$$\gamma_{xy} = \frac{2(1 + \nu)}{E} \tau_{xy}$$

$$(7.29)$$

and the equilibrium equations are reduced to two as in the case of plane strain,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$
(7.30)

We notice that Eqs. (7.29) and (7.30) are identical in form to Eqs. (7.8) and (7.18). Therefore, all equations for plane stress can be obtained by the respective equations of plane strain, if the effective elastic constants $\overline{\nu}$ and \overline{E} are replaced by the actual ones ν and E. Thus, we have:

Elastic constants

$$\mu = G = \frac{E}{2(1+\nu)}$$
(7.31a)

$$\lambda^{\star} = \frac{\nu E}{1 - \nu^2} \tag{7.31b}$$

where λ^* plays the role of the Lamé constant.

Constitutive equations

$$\{\varepsilon\} = [S]\{\sigma\} \tag{7.32}$$

$$\{\sigma\} = [C]\{\varepsilon\} \tag{7.33}$$

$$[S] = \frac{1}{E} \begin{vmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{vmatrix}$$
(7.34)
$$\begin{bmatrix} 1 & \nu & 0 \\ \end{bmatrix}$$

.

$$[C] = \frac{E}{1 - \nu^2} \begin{vmatrix} \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{vmatrix}$$
(7.35)

Navier's equations of equilibrium

$$\left. \nabla^{2} u + \frac{1+\nu}{1-\nu} \left(\frac{\partial^{2} u}{\partial x^{2}} + \frac{\partial^{2} v}{\partial x \partial y} \right) + \frac{1}{G} b_{x} = 0 \\ \nabla^{2} v + \frac{1+\nu}{1-\nu} \left(\frac{\partial^{2} u}{\partial x \partial y} + \frac{\partial^{2} v}{\partial y^{2}} \right) + \frac{1}{G} b_{y} = 0$$

$$(7.36)$$

Boundary tractions

$$t_{x} = \lambda^{*} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_{x} + \mu \left(\frac{\partial u}{\partial x} n_{x} + \frac{\partial v}{\partial x} n_{y} \right) + \mu \frac{\partial u}{\partial n}$$

$$t_{y} = \lambda^{*} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_{y} + \mu \left(\frac{\partial u}{\partial y} n_{x} + \frac{\partial v}{\partial y} n_{y} \right) + \mu \frac{\partial v}{\partial n}$$

$$(7.37)$$

Initial stresses due to temperature variation

$$\{\sigma_0\} = \frac{E \alpha \Delta T}{1 - \nu} \begin{cases} 1\\ 1\\ 0 \end{cases}$$
(7.38)

7.3 Betti's reciprocal identity

The derivation of the integral representation of the solution for two-dimensional elasticity problems requires the establishment of a reciprocal identity for Navier's operator, Eq. (7.19), similar to that of Green's for the Laplace operator, Eq. (2.16). Betti's reciprocal identity plays this role. It can be readily derived from the known Betti's theorem for the reciprocity of works, which is valid for the linear theory of

elasticity. For this purpose, we consider an elastic body occupying the volume V of the three-dimensional space and bounded by the surface S. Moreover, we consider two states of stress due to two different distributions of body forces and boundary quantities (displacements and tractions). We designate the displacements, body forces and tractions for the two states of stress as follows:

State I:

$$\mathbf{u} = \begin{cases} u \\ v \\ w \end{cases}, \quad \mathbf{b} = \begin{cases} b_x \\ b_y \\ b_z \end{cases}, \quad \mathbf{t} = \begin{cases} t_x \\ t_y \\ t_z \end{cases}$$
(7.39)

State II:

$$\mathbf{u}^{*} = \begin{cases} u^{*} \\ v^{*} \\ w^{*} \end{cases}, \quad \mathbf{b}^{*} = \begin{cases} b^{*}_{x} \\ b^{*}_{y} \\ b^{*}_{z} \end{cases}, \quad \mathbf{t}^{*} = \begin{cases} t^{*}_{x} \\ t^{*}_{y} \\ t^{*}_{z} \end{cases}$$
(7.40)

According to Betti's theorem, the work produced by the displacements of state (1) and the forces of state (11) is equal to the work produced by the displacements of state (11) and the forces of state (1). This may be expressed as

$$\int_{V} \mathbf{u} \cdot \mathbf{b}^{*} \, dV + \int_{S} \mathbf{u} \cdot \mathbf{t}^{*} \, dS = \int_{V} \mathbf{u}^{*} \cdot \mathbf{b} \, dV + \int_{S} \mathbf{u}^{*} \cdot \mathbf{t} \, dS$$
(7.41)

or using Eqs. (7.39) and (7.40)

$$\int_{V} \left(u \, b_{x}^{*} + v \, b_{y}^{*} + w \, b_{z}^{*} \right) dV + \int_{S} \left(u \, t_{x}^{*} + v \, t_{y}^{*} + w \, t_{z}^{*} \right) dS$$
$$= \int_{V} \left(u^{*} \, b_{x} + v^{*} \, b_{y} + w^{*} \, b_{z} \right) dV + \int_{S} \left(u^{*} \, t_{x} + v^{*} \, t_{y} + w^{*} \, t_{z} \right) dS \qquad (7.42)$$

For the plane problem we distinguish the following two cases:

(a) Plain strain. We consider the part of the cylindrical body cut by the two planes z and z + 1, that is a slice of unit thickness. In this case, it is $b_z = 0$, $b_z^* = 0$ inside the body, $t_z = 0$, $t_z^* = 0$ on the cylindrical surface and by virtue of Eqs. (7.22) or (7.23) it is $t_x = t_y = 0$, $t_x^* = t_y^* = 0$ on the plane sections. Moreover, taking into account that w = c, $w^* = c^*$ on the plane sections and, thus, the works produced on them are of equal magnitude but opposite sign, Eq. (7.42) becomes

$$\int_{\Omega} \left(u \, b_x^* + v \, b_y^* \right) d\Omega + \int_{\Gamma} \left(u \, t_x^* + v \, t_y^* \right) ds$$
$$= \int_{\Omega} \left(u^* \, b_x + v^* \, b_y \right) d\Omega + \int_{\Gamma} \left(u^* \, t_x + v^* \, t_y \right) ds \tag{7.43}$$

(b) *Plane stress.* We consider a thin plate of constant thickness h. In this case, it is $b_z = 0$, $b_z^* = 0$ inside the body and $t_z = 0$, $t_z^* = 0$ on the whole surface S and by virtue of Eqs. (7.37) it is $t_x = t_y = 0$, $t_x^* = t_y^* = 0$ on the plane sections. Hence, also Eq. (7.42) takes the form of Eq. (7.43) when applied to this body.

Furthermore, if the body forces in Eq. (7.43) are replaced by their expressions from Eqs. (7.19), we obtain the reciprocal identity for the Navier operator

$$\int_{\Omega} \left\{ \left[u \, N_x(u^*, v^*) + v \, N_y(u^*, v^*) \right] - \left[u^* \, N_x(u, v) + v^* \, N_y(u, v) \right] \right\} d\Omega$$
$$= -\int_{\Gamma} \left[\left(u \, t_x^* + v \, t_y^* \right) - \left(u^* \, t_x + v^* \, t_y \right) \right] ds \tag{7.44}$$

where the operators $N_x(.,.)$ and $N_y(.,.)$ are defined on the basis of Eqs. (7.19) as:

$$N_{x}(u,v) = -G\left[\nabla^{2}u + \frac{1+\overline{\nu}}{1-\overline{\nu}}\left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}v}{\partial x\partial y}\right)\right]$$

$$N_{y}(u,v) = -G\left[\nabla^{2}v + \frac{1+\overline{\nu}}{1-\overline{\nu}}\left(\frac{\partial^{2}u}{\partial x\partial y} + \frac{\partial^{2}v}{\partial y^{2}}\right)\right]$$
(7.45)

Equations (7.45) are valid for plane strain. We recall that for plane stress $\overline{\nu}$ should be replaced by ν .

7.4 Fundamental solution

In order to derive the boundary integral equations pertaining to the plane elastostatic problem, it is necessary to establish first the fundamental solution of the Navier Eqs. (7.19). From the physical point of view, the fundamental solution expresses the displacements produced in an infinite plane body by a concentrated unit body force. This solution is attributed to Kelvin and this is why it is known in the literature as *Kelvin's solution*. It can be established using the procedure described below.

Consider the concentrated force $\mathbf{F}(F_{\xi}, F_{\eta})$, $|\mathbf{F}| = 1$, applied at point $Q(\xi, \eta)$ of the plane (see Fig. 7.4). It is apparent that the components F_{ξ} and F_{η} of the force \mathbf{F} are the direction cosines of the unit vector representing this force. The density of the body forces produced by the force \mathbf{F} at a point P(x, y) can be represented using the delta function. Thus, we have:



Figure 7.4 Concentrated unit force $\mathbf{F}(F_{\xi}, F_{\eta})$ applied at point $Q(\xi, \eta)$ of the plane.

$$\mathbf{b} = \delta(P - Q) \mathbf{F} \tag{7.46}$$

or

$$b_{x} = \delta(P - Q) F_{\xi}$$

$$b_{y} = \delta(P - Q) F_{\eta}$$

$$(7.47)$$

In this case, Eqs. (7.19) are written as

$$\nabla^{2}u + \frac{1+\overline{\nu}}{1-\overline{\nu}} \left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}v}{\partial x\partial y} \right) + \frac{1}{G} \delta(P-Q) F_{\xi} = 0$$

$$\nabla^{2}v + \frac{1+\overline{\nu}}{1-\overline{\nu}} \left(\frac{\partial^{2}u}{\partial x\partial y} + \frac{\partial^{2}v}{\partial y^{2}} \right) + \frac{1}{G} \delta(P-Q) F_{\eta} = 0$$

$$(7.48)$$

The fundamental solution for the Navier operator is a singular solution of Eqs. (7.48), which can be established by expressing the displacement components in terms of the *Galerkin functions*. Thus, we set

$$2G u = \frac{2}{1+\overline{\nu}} \nabla^2 \phi - \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \\ \cdot \\ 2G v = \frac{2}{1+\overline{\nu}} \nabla^2 \psi - \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right)$$
(7.49)

where $\phi = \phi(x, y)$ and $\psi = \psi(x, y)$ are the Galerkin functions. They represent the components of a vector, the so-called *Galerkin vector* [3].

Introducing Eqs. (7.49) into the equilibrium equations (7.48), we arrive at

$$\nabla^{4}\phi = -(1+\overline{\nu}) \,\delta(P-Q) F_{\xi}
 \nabla^{4}\psi = -(1+\overline{\nu}) \,\delta(P-Q) F_{\eta}$$
(7.50)

where

$$\nabla^4 = \nabla^2 \, \nabla^2 = \frac{\partial^4}{\partial x^4} + 2 \, \frac{\partial^2}{\partial x^2} \, \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}$$

is the biharmonic operator.

Hence, Eqs. (7.49) are a solution to Eqs. (7.48), if the functions ϕ and ψ represent singular particular solutions of Eqs. (7.50). These solutions can be established by working as follows:

The first of Eqs. (7.50) is written as

$$\nabla^2 \Phi = -(1+\vec{\nu}) \,\delta(P-Q) F_{\xi} \tag{7.51}$$

where it was set

$$\nabla^2 \phi = \Phi \tag{7.52}$$

Equation (7.51) has the form of Eq. (3.8), Thus, a singular particular solution of this equation is

$$\Phi = -\frac{(1+\overline{\nu})}{2\pi} \left(\ell nr + B \right) F_{\xi}$$

where r = |P - Q| and B is an arbitrary constant. Consequently, Eq. (7.52) becomes

$$\nabla^2 \phi = -\frac{(1+\overline{\nu})}{2\pi} \left(\ell nr + B \right) F_{\xi}$$
(7.53)

Since the solution is independent of the polar angle at the source point Q, one can write for $r \neq 0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\frac{(1+\overline{\nu})}{2\pi}\left(\ell nr + B\right)F_{\xi}$$

Two consecutive integrations result in

$$\phi = -\frac{(1+\overline{\nu})}{2\pi} F_{\xi} \left[\frac{r^2}{4} \ln r + \frac{1}{4} (B-1)r^2 + C \ln r + D \right]$$
(7.54)

where C and D are arbitrary constants. Since we are interested for a particular solution, we can set B = 1 and C = D = 0 so that the solution has the simplest form. Thus, we have

$$\phi = -\frac{(1+\overline{\nu})}{8\pi} F_{\xi} r^2 \ell nr$$
(7.55)

Similarly, we obtain

$$\psi = -\frac{(1+\overline{\nu})}{8\pi} F_{\eta} r^2 \ell nr$$
(7.56)

Substitution of Eqs. (7.55) and (7.56) into expressions (7.49) yields the fundamental solution for the Navier equations. As we will see in the next section, the fundamental solution for (i) $F_{\xi} = 1$, $F_{\eta} = 0$ and (ii) $F_{\xi} = 0$, $F_{\eta} = 1$ are required for deriving the boundary integral equations. Therefore, the derivation of the expressions for the fundamental solution is limited to these two cases.

(i)
$$F_{\xi} = 1$$
, $F_{\eta} = 0$

We readily have

$$\phi = -\frac{1+\overline{\nu}}{8\pi} r^{2} \ell n r$$

$$\psi = 0$$

$$\nabla^{2} \phi = -\frac{1+\overline{\nu}}{2\pi} (\ell n r + 1)$$

$$\frac{\partial^{2} \phi}{\partial x^{2}} = -\frac{1+\overline{\nu}}{8\pi} (2 \ell n r + 2 r_{x}^{2} + 1)$$

$$\frac{\partial^{2} \phi}{\partial x \partial y} = -\frac{1+\overline{\nu}}{8\pi} 2 r_{x} r_{y}$$

$$(7.57)$$

In the previous expressions and in what it follows r_{x} and r_{y} express the derivatives of the distance r with respect to x and y, respectively, and are given as (see Appendix A)

$$r_{,x} = -rac{\xi - x}{r}, \quad r_{,y} = -rac{\eta - y}{r}$$

which obviously satisfy the relation $r_{x}^{2} + r_{y}^{2} = 1$.

Equations (7.57) are substituted into expressions (7.49) to yield

$$U_{x\xi} = -\frac{1}{8\pi G} \left[(3 - \overline{\nu}) \ell nr - (1 + \overline{\nu}) r_{,x}^{2} + \frac{7 - \overline{\nu}}{2} \right]$$

$$U_{y\xi} = \frac{1}{8\pi G} (1 + \overline{\nu}) r_{,x} r_{,y}$$
(7.58)

where $U_{x\xi}$ and $U_{y\xi}$ represent the displacements u and v, respectively. The first subscript denotes the direction of the displacement, whereas the second one the direction of the unit force (see Fig. 7.5).

(ii)
$$F_{\xi} = 0$$
, $F_{\eta} = 1$

Following the same procedure as in case (i), we obtain

$$\phi = 0$$

$$\psi = -\frac{1+\overline{\nu}}{8\pi} r^{2} \ell n r$$

$$\nabla^{2} \psi = -\frac{1+\overline{\nu}}{2\pi} (\ell n r + 1)$$

$$\frac{\partial^{2} \psi}{\partial x \partial y} = -\frac{1+\overline{\nu}}{8\pi} 2 r_{x} r_{y}$$

$$\frac{\partial^{2} \psi}{\partial y^{2}} = -\frac{1+\overline{\nu}}{8\pi} (2 \ell n r + 2 r_{y}^{2} + 1)$$
(7.59)

and substituting Eqs. (7.59) into the Eqs. (7.49), we find

$$U_{x\eta} = \frac{1}{8\pi G} (1 + \overline{\nu}) r_{,x} r_{,y}$$

$$U_{y\eta} = -\frac{1}{8\pi G} \left[(3 - \overline{\nu}) \ell nr - (1 + \overline{\nu}) r_{,y}^{2} + \frac{7 - \overline{\nu}}{2} \right]$$
(7.60)

Employing indicial notation for the coordinates of points P and Q, i.e. x_1 , x_2 and ξ_1 , ξ_2 , respectively, Eqs. (7.58) and (7.60) may be written as

$$U_{ij} = -\frac{1}{8\pi G} \left[C_1 \,\delta_{ij} \,\ell nr - C_2 \,r_{,i} \,r_{,j} + \delta_{ij} \,C_3 \right]$$
(7.61)

where



Figure 7.5 Component of the two-dimensional elasticity fundamental solution.

$$C_1 = 3 - \bar{\nu}, \quad C_2 = 1 + \bar{\nu}, \quad C_3 = \frac{7 - \bar{\nu}}{2}$$
 (7.62)

The constant C_3 can be omitted, because, as it will be shown in Section 7.12, it produces only rigid body displacements, which do not influence the stresses and the strains.

Equation (7.61) may also be written in matrix form as

$$\mathbf{U}(P,Q) = \begin{bmatrix} U_{x\xi} & U_{x\eta} \\ U_{y\xi} & U_{y\eta} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$
(7.63)

The components of the fundamental solution represent the components of a *two-point* second order tensor [3, 4] known as *Green's tensor* [5].

Since it is (see Appendix A)

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$
, $r_x = -r_{\xi}$, $r_y = -r_{\eta}$

one can easily find out that the tensor defined above is symmetric with respect to points P and Q. This means that its components do not change, if the role of the two points is interchanged. Namely, P becomes the point where the force is applied (source point) and Q the point where the displacement is evaluated (field point). Thus, it is

$$\mathbf{U}(P,Q) = \mathbf{U}(Q,P)$$

or

$$\begin{bmatrix} U_{x\xi} & U_{x\eta} \\ U_{y\xi} & U_{y\eta} \end{bmatrix} = \begin{bmatrix} U_{\xi x} & U_{\xi y} \\ U_{\eta x} & U_{\eta y} \end{bmatrix}$$
(7.64)

This symmetry expresses the Betti-Maxwell law for the *reciprocity of the displace*ments.

7.5 Stresses due to a unit concentrated force

The expressions for the stress components due to the unit force can be evaluated using Eqs. (7.3). We distinguish again two cases:

(i)
$$F_{\xi} = 1, \ F_{\eta} = 0$$

 $\sigma_{x\xi} = \lambda (U_{x\xi,x} + U_{y\xi,y}) + 2\mu U_{x\xi,x}$
 $\sigma_{y\xi} = \lambda (U_{x\xi,x} + U_{y\xi,y}) + 2\mu U_{y\xi,y}$
 $\tau_{xy\xi} = \mu (U_{x\xi,y} + U_{y\xi,x})$
(7.65)

Substituting Eqs. (7.58) into the above equations, yields

$$\sigma_{x\xi} = \frac{A_{1}}{r} \left(A_{2} r_{x} + 2 r_{x}^{3} \right)$$

$$\sigma_{y\xi} = \frac{A_{1}}{r} \left(-A_{2} r_{x} + 2 r_{x} r_{y}^{2} \right)$$

$$\tau_{xy\xi} = \frac{A_{1}}{r} \left(A_{2} r_{y} + 2 r_{x}^{2} r_{y} \right)$$
(7.66)

where

$$A_1 = -\frac{1+\overline{\nu}}{4\pi}, \quad A_2 = \frac{1-\overline{\nu}}{1+\overline{\nu}}$$
 (7.67)

(ii) $F_{\xi} = 0$, $F_{\eta} = 1$

$$\sigma_{x\eta} = \lambda \left(U_{x\eta,x} + U_{y\eta,y} \right) + 2\mu U_{x\eta,x}$$

$$\sigma_{y\eta} = \lambda \left(U_{x\eta,x} + U_{y\eta,y} \right) + 2\mu U_{y\eta,y}$$

$$\tau_{xy\eta} = \mu \left(U_{x\eta,y} + U_{y\eta,x} \right)$$
(7.68)

Substituting Eqs. (7.60) into the above equations, results in

$$\sigma_{x\eta} = \frac{A_1}{r} \left(-A_2 r_{,y} + 2 r_{,x}^2 r_{,y} \right)$$

$$\sigma_{y\eta} = \frac{A_1}{r} \left(A_2 r_{,y} + 2 r_{,y}^3 \right)$$

$$\tau_{xy\eta} = \frac{A_1}{r} \left(A_2 r_{,x} + 2 r_{,x} r_{,y}^2 \right)$$
(7.69)

.

If indicial notation is employed for the coordinates, i.e. x_1 , x_2 , then Eqs. (7.66) and (7.69) are reduced to [6]

$$\sigma_{ijk} = \frac{A_1}{r} \left[A_2 \left(\delta_{ik} r_{,j} + \delta_{jk} r_{,i} - \delta_{ij} r_{,k} \right) + 2 r_{,i} r_{,j} r_{,k} \right] \quad (i, j, k = 1, 2)$$
(7.70)

The subscript k = 1, 2 pertains to the directions ξ and η of the unit force, respectively. It should be noted that $\sigma_{11} = \sigma_x$, $\sigma_{22} = \sigma_y$, $\sigma_{12} = \tau_{xy}$.

7.6 Boundary tractions due to a unit concentrated force

Expressions for the tractions on the boundary due to the concentrated force can be derived using Eq. (7.22). We distinguish two cases:

(i)
$$F_{\xi} = 1$$
, $F_{\eta} = 0$
 $T_{x\xi} = \sigma_{x\xi} n_x + \tau_{xy\xi} n_y$
 $T_{y\xi} = \tau_{xy\xi} n_x + \sigma_{y\xi} n_y$

or using Eqs. (7.66)

$$T_{x\xi} = \frac{A_{1}}{r} \left(A_{2} + 2r_{x}^{2} \right) r_{n}$$

$$T_{y\xi} = \frac{A_{1}}{r} \left(2r_{x}r_{y}r_{n} + A_{2}r_{t} \right)$$
(7.71)

(ii) $F_{\xi} = 0$, $F_{\eta} = 1$

$$T_{x\eta} = \sigma_{x\eta} \, n_x + \tau_{xy\eta} \, n_y$$

$$T_{y\eta} = \tau_{xy\eta} \, n_x + \sigma_{y\eta} \, n_y$$

or using Eqs. (7.69)

$$T_{x\eta} = \frac{A_{1}}{r} \left(2 r_{,x} r_{,y} r_{,n} - A_{2} r_{,t} \right)$$

$$T_{y\eta} = \frac{A_{1}}{r} \left(A_{2} + 2 r_{,y}^{2} \right) r_{,n}$$
(7.72)

where $r_{,n} = r_{,x}n_x + r_{,y}n_y$ expresses the derivative of r in the direction of the outward normal to the curve (boundary) passing through the point (x, y) and $r_{,t} = -r_{,x}n_y + r_{,y}n_x$ the derivative along the tangent to this curve. The vectors **n** and **t** define a right handed system of axes.

Using indicial notation Eqs. (7.71) and (7.72) are reduced to [6]

$$T_{ik} = \frac{A_1}{r} \left[\left(A_2 \,\delta_{ik} + 2 \,r_{,i} \,r_{,k} \,\right) r_{,n} + A_2 \left(r_{,i} n_k - r_{,k} n_i \,\right) \right]$$
(7.73)
(*i*, *k* = 1, 2)

7.7 Integral representation of the solution

The integral representation of the solution for the two-dimensional elastostatic problem is derived from the reciprocal identity (7.43) by considering as state (II) the state of stress produced by a unit body force at point Q. The unit force is applied first in the ξ -direction and then in the η -direction.

(i)
$$F_{\mathcal{E}} = 1$$
, $F_{\eta} = 0$

For this case, the state (II) inside the domain Ω is defined as

$$b_x^* = \delta(P-Q), \qquad b_y^* = 0$$

$$u^* = U_{x\xi}(P,Q), \quad v^* = U_{y\xi}(P,Q)$$

while on the boundary Γ it is

$$t_x^* = T_{x\xi}(p,Q), \qquad t_y^* = T_{y\xi}(p,Q)$$

where $P \in \Omega$ and $p \in \Gamma$.

Introducing the above expressions into Eq. (7.43) and taking into account that

$$\int_{\Omega} u \, b_x^* \, d\Omega = \int_{\Omega} u(P) \, \delta(P-Q) \, d\Omega_P = u(Q)$$

we obtain the integral representation of the solution for the displacement in the x – direction at points Q inside the domain Ω in the following form

$$u(Q) = \int_{\Omega} \Big[U_{x\xi}(P,Q) b_x(P) + U_{y\xi}(P,Q) b_y(P) \Big] d\Omega_P + \int_{\Gamma} \Big[U_{x\xi}(p,Q) t_x(p) + U_{y\xi}(p,Q) t_y(p) \Big] ds_p - \int_{\Gamma} \Big[T_{x\xi}(p,Q) u(p) + T_{y\xi}(p,Q) v(p) \Big] ds_p$$
(7.74)

(ii) $F_{\xi} = 0$, $F_{\eta} = 1$

For this case, inside the domain Ω it is

$$b_x^* = 0, \qquad b_y^* = \delta(P - Q)$$

$$u^* = U_{x\eta}(P,Q), \quad v^* = U_{y\eta}(P,Q)$$

and on the boundary Γ

$$t_x^* = T_{x\eta}(p,Q), \qquad t_y^* = T_{y\eta}(p,Q)$$

Introducing the above expressions into Eq. (7.43) and taking into account that

$$\int_{\Omega} v \, b_y^* \, d\Omega = \int_{\Omega} v(P) \, \delta(P-Q) \, d\Omega_P = v(Q)$$

we find

$$v(Q) = \int_{\Omega} \left[U_{x\eta}(P,Q) b_{x}(P) + U_{y\eta}(P,Q) b_{y}(P) \right] d\Omega_{P} + \int_{\Gamma} \left[U_{x\eta}(p,Q) t_{x}(p) + U_{y\eta}(p,Q) t_{y}(p) \right] ds_{p} - \int_{\Gamma} \left[T_{x\eta}(p,Q) u(p) + T_{y\eta}(p,Q) v(p) \right] ds_{p}$$
(7.75)

Equations (7.74) and (7.75) represent the integral representation of the solution to plane elasticity Navier's equations. We notice that the role of points P(x, y) and $Q(\xi, \eta)$ in these equations has been interchanged due to reciprocity. Thus, point $Q(\xi, \eta)$ is now the field point and P(x, y) the source point. The vector **n** in the expressions (7.71) and (7.72) of $T_{x\xi}$, $T_{y\xi}$, $T_{x\eta}$ and $T_{y\eta}$ is normal to the boundary at the point $p \in \Gamma$ where the load is applied. For reasons of consistency, the initial notation is restored. Hence, $P \in \Omega$ or $p \in \Gamma$ will designate field points, while $Q \in \Omega$ or $q \in \Gamma$ will designate source points, i.e. points where unit forces are applied. Thus, Eqs. (7.74) and (7.75) may be rewritten under this notation and at the same time may be combined in a single matrix equation as

$$\begin{cases} u(P) \\ v(P) \end{cases} = \int_{\Omega} \begin{bmatrix} U_{\xi x}(Q, P) & U_{\eta x}(Q, P) \\ U_{\xi y}(Q, P) & U_{\eta y}(Q, P) \end{bmatrix} \begin{bmatrix} b_{\xi}(Q) \\ b_{\eta}(Q) \end{bmatrix} d\Omega_{Q}$$

$$+ \int_{\Gamma} \begin{bmatrix} U_{\xi x}(q, P) & U_{\eta x}(q, P) \\ U_{\xi y}(q, P) & U_{\eta y}(q, P) \end{bmatrix} \begin{bmatrix} t_{\xi}(q) \\ t_{\eta}(q) \end{bmatrix} ds_{q}$$

$$- \int_{\Gamma} \begin{bmatrix} T_{\xi x}(q, P) & T_{\eta x}(q, P) \\ T_{\xi y}(q, P) & T_{\eta y}(q, P) \end{bmatrix} \begin{bmatrix} u(q) \\ v(q) \end{bmatrix} ds_{q}$$

$$(7.76)$$

This change in notation necessitates the replacement of derivatives $r_{,x}$ and $r_{,y}$ in Eqs. (7.58), (7.60), (7.71) and (7.72) with $r_{,\xi}$ and $r_{,\eta}$, respectively, without however changing their sign. Thus, we have

$$U_{\xi x} = -\frac{1}{8\pi G} \left[(3 - \overline{\nu}) \ell n r - (1 + \overline{\nu}) r_{\xi}^{2} + \frac{7 - \overline{\nu}}{2} \right]$$

$$U_{\eta x} = U_{\xi y} = \frac{1}{8\pi G} (1 + \overline{\nu}) r_{\xi} r_{\eta}$$

$$U_{\eta y} = -\frac{1}{8\pi G} \left[(3 - \overline{\nu}) \ell n r - (1 + \overline{\nu}) r_{\eta}^{2} + \frac{7 - \overline{\nu}}{2} \right]$$

$$T_{\xi x} = \frac{A_{1}}{r} \left(A_{2} + 2 r_{\xi}^{2} \right) r_{\eta}$$

$$T_{\eta x} = \frac{A_{1}}{r} \left(2 r_{\xi} r_{\eta} r_{\eta} + A_{2} r_{\eta} \right)$$

$$T_{\xi y} = \frac{A_{1}}{r} \left(2 r_{\xi} r_{\eta} r_{\eta} - A_{2} r_{\eta} \right)$$

$$T_{\eta y} = \frac{A_{1}}{r} \left(A_{2} + 2 r_{\eta}^{2} \right) r_{\eta}$$

$$(7.77)$$

where $r_{,n} = r_{,\xi} n_x + r_{,\eta} n_y$ and $r_{,t} = -r_{,\xi} n_y + r_{,\eta} n_x$.

Employing indicial notation for the two directions, i.e. x_1 , x_2 , Eq. (7.76) is reduced to [6]

$$u_i(P) = \int_{\Omega} U_{ji} b_j d\Omega + \int_{\Gamma} \left(U_{ji} t_j - T_{ji} u_j \right) ds$$

$$(i, j = 1, 2)$$
(7.78)

7.8 Boundary integral equations

As in the case of the Laplace equation discussed in Chapter 3, the boundary integral equations for the problem at hand are produced by letting point $P \in \Omega$ in Eqs. (7.76) move to a point $p \in \Gamma$. We notice from Eqs. (7.58), (7.60), (7.71) and (7.72) that the kernels in the boundary integrals of Eqs. (7.76) are singular, that is they exhibit a singular behavior when $P \in \Omega \rightarrow p \in \Gamma$, because $r \rightarrow 0$ when $p \rightarrow q$. Therefore, it is necessary to examine the behavior of the boundary integrals in Eqs. (7.76), when we let point $P \in \Omega$ coincide with a point $p \in \Gamma$.



Figure 7.6 Geometric definitions related to a corner point *P* of a non-smooth boundary.

We examine the general case of a non-smooth boundary, and we assume that $P \equiv p$ is a corner point (see Fig. 7.6). Next we consider the domain Ω^* , which results from Ω by subtracting a small circular sector with center P, radius ε and confined by the arcs PA and PB. We denote the circular arc AB by Γ_{ε} and the sum of the arcs AP and PB by ℓ . The outward normal to Γ_{ε} coincides with the radius. Moreover, α is the angle between the two tangents to the boundary at point P. Obviously, it is

$$\ell_{\varepsilon \to 0} \Gamma_{\varepsilon} = 0$$

 $\ell_{\varepsilon \to 0} (\Gamma - \ell) = \Gamma$

and the chords PA and PB become tangents to the boundary at point P for $\varepsilon \rightarrow 0$.

Using the aforementioned notation, we apply the reciprocity relation (7.43) for the domain Ω^* with (a) $b_x^* = \delta(Q - P)$, $b_y^* = 0$ and (b) $b_x^* = 0$, $b_y^* = \delta(Q - P)$, where $Q \in \Omega^*$, $P \in \overline{\Omega}^* \equiv \Omega - \Omega^*$. In both cases the unit load is applied at point P(x, y). Since point P lies outside the domain, the domain integrals in the left-hand side of Eq. (7.43) vanish. By utilizing indicial notation, we can write

$$\int_{\Gamma-\ell} T_{ji}(P,q) u_j(q) ds_q + \int_{\Gamma_\ell} T_{ji}(P,q) u_j(q) ds_q$$

$$= \int_{\Omega} U_{ji}(P,Q) b_j d\Omega_Q + \int_{\Gamma-\ell} U_{ji}(P,q) t_j(q) ds_q + \int_{\Gamma_\ell} U_{ji}(P,q) t_j(q) ds_q + \int_{\Gamma_\ell} U_{ji}(P,q) t_j(q) ds_q$$
(7.79)

We will examine the behavior of the integrals in Eq. (7.79) for $\varepsilon \to 0$. It is apparent that, in the limit, line integrals on $\Gamma - \ell$ become integrals on Γ , while domain integrals over Ω^* become integrals over Ω . Therefore, it remains to examine the behavior of integrals on Γ_{ε} . Namely,

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} U_{ji} t_j \, ds \tag{7.80}$$

and

$$\lim_{s \to 0} \int_{\Gamma_{i}} T_{ji} \, u_{j} \, ds \tag{7.81}$$

We examine first the integrals (7.80) whose kernels are the fundamental solution U_{μ} .

Using the mean value theorem of integral calculus, the integrals (7.80) can be written in expanded form as

$$t_{\xi}(q^{*}) \underset{\varepsilon \to 0}{\ell im} \int_{\Gamma_{\varepsilon}} U_{\xi x} ds + t_{\eta}(q^{*}) \underset{\varepsilon \to 0}{\ell im} \int_{\Gamma_{\varepsilon}} U_{\eta x} ds$$

$$t_{\xi}(q^{*}) \underset{\varepsilon \to 0}{\ell im} \int_{\Gamma_{\varepsilon}} U_{\xi y} ds + t_{\eta}(q^{*}) \underset{\varepsilon \to 0}{\ell im} \int_{\Gamma_{\varepsilon}} U_{\eta y} ds$$

$$(7.82)$$

where q^* is a point on Γ_{ε} , which is generally different for each of the four terms. Obviously it is $q^* \to p \equiv P$, when $\varepsilon \to 0$.

On the basis of Eqs. (7.58) and (7.60), we note that the integrals in Eqs. (7.82) involve terms of the following forms

$$I_1 = \int_{\Gamma_{\varepsilon}} \ell nr \, ds \,, \quad I_2 = \int_{\Gamma_{\varepsilon}} r_{\xi}^2 \, ds \,, \quad I_3 = \int_{\Gamma_{\varepsilon}} r_{\xi} r_{\eta} \, ds \,, \quad I_4 = \int_{\Gamma_{\varepsilon}} r_{\eta}^2 \, ds$$

As it was shown in Section 3.3, it is

$$I_{1} = \int_{\Gamma_{\varepsilon}} \ell nr \, ds = \int_{\theta_{1}}^{\theta_{2}} \varepsilon \, \ell n\varepsilon \, d(-\theta) = \varepsilon \, \ell n\varepsilon \, (\theta_{1} - \theta_{2})$$

which in the limit yields

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \ln r \, ds = (\theta_1 - \theta_2) \lim_{\varepsilon \to 0} (\varepsilon \ln \varepsilon) = 0$$

and consequently

$$I_1 = 0$$

It remains to find the limits of the other three integrals. We can write (see Appendix A)

$$r_{\ell} = \cos \theta$$
 and $r_{\eta} = \sin \theta$

Hence,

$$I_{2} = \int_{\Gamma_{\varepsilon}} r_{\xi}^{2} ds = \int_{\theta_{1}}^{\theta_{2}} \cos^{2} \theta \varepsilon d(-\theta) = -\varepsilon \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{\theta_{1}}^{\theta_{2}}$$
$$I_{3} = \int_{\Gamma_{\varepsilon}} r_{\xi} r_{\eta} ds = \int_{\theta_{1}}^{\theta_{2}} \cos \theta \sin \theta \varepsilon d(-\theta) = \varepsilon \left[\frac{\cos 2\theta}{4} \right]_{\theta_{1}}^{\theta_{2}}$$
$$I_{4} = \int_{\Gamma_{\varepsilon}} r_{\eta}^{2} ds = \int_{\theta_{1}}^{\theta_{2}} \sin^{2} \theta \varepsilon d(-\theta) = -\varepsilon \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_{\theta_{1}}^{\theta_{2}}$$

and consequently

$$\lim_{\varepsilon \to 0} I_2 = 0, \quad \lim_{\varepsilon \to 0} I_3 = 0, \quad \lim_{\varepsilon \to 0} I_4 = 0$$

Thus, the last integral of Eq. (7.80) vanishes for $\varepsilon \to 0$ and the remaining three integrals in the right-hand side of Eq. (7.79) vary continuously as point $P \in \Omega$ approaches point $p \in \Gamma$.

Using again the mean value theorem of integral calculus, the integrals (7.81) with kernels T_{ji} can be written in expanded form as

$$\left. \begin{array}{c} u(q^{*}) \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\xi x} \, ds + v(q^{*}) \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\eta x} \, ds \\ u(q^{*}) \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\xi y} \, ds + v(q^{*}) \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\eta y} \, ds \end{array} \right\}$$

$$(7.83)$$

where q^* is a point on Γ_{ε} , generally different in each of the four terms.

Referring to Fig. 7.6, we note that $\phi = \text{angle}(\mathbf{r}, \mathbf{n}) = \pi$, where $r \equiv \varepsilon$. Therefore,

 $r_{n} = \cos \phi = -1$ and $r_{t} = \sin \phi = 0$

Consequently, using Eqs. (7.71), (7.72) and (7.67) and taking into account that $ds = r d(-\theta) = -\varepsilon d\theta$ (see Fig. 7.6 and Section 3.3), we obtain

$$\varepsilon_{\xi x} = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\xi x} \, ds = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \frac{A_1}{r} \Big(A_2 + 2 r_{\xi}^2 \Big) r_n \, ds$$
$$= \frac{1 + \overline{\nu}}{8\pi} \Big(\frac{4}{1 + \overline{\nu}} \left[\theta \right]_{\theta_2}^{\theta_1} + [\sin 2\theta]_{\theta_2}^{\theta_1} \Big)$$
(7.84)

$$\varepsilon_{\eta x} = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\eta x} \, ds = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \frac{A_1}{r} \Big(2 r_{\xi} r_{\eta} r_{\eta} + A_2 r_{\eta} \Big) ds$$
$$= -\frac{1 + \overline{\nu}}{8\pi} \left[\cos 2\theta \right]_{\theta_2}^{\theta_1}$$
(7.85)

$$\varepsilon_{\xi y} = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\xi y} \, ds = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \frac{A_1}{r} \Big(2 r_{\xi} r_{\eta} r_{\eta} - A_2 r_{t} \Big) ds$$
$$= -\frac{1 + \overline{\nu}}{8\pi} [\cos 2\theta]_{\theta_2}^{\theta_1}$$
(7.86)

$$\varepsilon_{\eta y} = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} T_{\eta y} \, ds = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \frac{A_1}{r} \Big(A_2 + 2 r_{\eta}^2 \Big) r_{\eta} \, ds$$
$$= \frac{1 + \overline{\nu}}{8\pi} \left(\frac{4}{1 + \overline{\nu}} [\theta]_{\theta_2}^{\theta_1} - [\sin 2\theta]_{\theta_2}^{\theta_1} \right)$$
(7.87)

Obviously, at points p where the boundary is smooth it is

$$[\theta]_{\theta_2}^{\theta_1} = \theta_1 - \theta_2 = \pi$$
$$[\cos 2\theta]_{\theta_2}^{\theta_1} = \cos 2\theta_1 - \cos 2\theta_2 = \cos 2(\theta_2 + \pi) - \cos 2\theta_2 = 0$$
$$[\sin 2\theta]_{\theta_2}^{\theta_1} = \sin 2\theta_1 - \sin 2\theta_2 = \sin 2(\theta_2 + \pi) - \sin 2\theta_2 = 0$$

and Eqs. (7.84) through (7.87) become

$$\left. \begin{array}{c} \varepsilon_{\xi x} = \frac{1}{2}, \quad \varepsilon_{\eta x} = 0 \\ \varepsilon_{\xi y} = 0, \quad \varepsilon_{\eta y} = \frac{1}{2} \end{array} \right\}$$

$$(7.88)$$

On the basis of the previous discussion, the boundary integral equations (7.79) may be written now in matrix form as

$$\begin{bmatrix} \varepsilon_{\xi x} & \varepsilon_{\eta x} \\ \varepsilon_{\xi y} & \varepsilon_{\eta y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \int_{\Omega} \begin{bmatrix} U_{\xi x} & U_{\eta x} \\ U_{\xi y} & U_{\eta y} \end{bmatrix} \begin{bmatrix} b_{\xi} \\ b_{\eta} \end{bmatrix} d\Omega + \int_{\Gamma} \begin{bmatrix} U_{\xi x} & U_{\eta x} \\ U_{\xi y} & U_{\eta y} \end{bmatrix} \begin{bmatrix} t_{\xi} \\ t_{\eta} \end{bmatrix} ds$$
$$- \int_{\Gamma} \begin{bmatrix} T_{\xi x} & T_{\eta x} \\ T_{\xi y} & T_{\eta y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ds$$
(7.89)

or using indicial notation

$$\varepsilon_{ji} u_j = \int_{\Omega} U_{ji} b_j d\Omega + \int_{\Gamma} \left(U_{ji} t_j - T_{ji} u_j \right) ds$$
(7.90)

Apparently, it is $\varepsilon_{ij} = \frac{1}{2} \delta_{ij}$ at points p where the boundary is smooth.

7.9 Integral representation of the stresses

The stress components σ_x , σ_y and τ_{xy} at a point P(x, y) inside the domain Ω are derived from Eqs. (7.3). Thus, introducing the displacement components u and v from Eqs. (7.76) yields

$$\sigma_{x} = \int_{\Omega} \left\{ \left[\lambda (U_{\xi x,x} + U_{\xi y,y}) + 2\mu U_{\xi x,x} \right] b_{\xi} + \left[\lambda (U_{\eta x,x} + U_{\eta y,y}) + 2\mu U_{\eta x,x} \right] b_{\eta} \right\} d\Omega + \int_{\Gamma} \left\{ \left[\lambda (U_{\xi x,x} + U_{\xi y,y}) + 2\mu U_{\xi x,x} \right] t_{\xi} + \left[\lambda (U_{\eta x,x} + U_{\eta y,y}) + 2\mu U_{\eta x,x} \right] t_{\eta} \right\} ds - \int_{\Gamma} \left\{ \left[\lambda (T_{\xi x,x} + T_{\xi y,y}) + 2\mu T_{\xi x,x} \right] u + \left[\lambda (T_{\eta x,x} + T_{\eta y,y}) + 2\mu T_{\eta x,x} \right] v \right\} ds$$
(7.91)

or setting

$$\sigma_{x\xi} = \lambda (U_{\xi x,x} + U_{\xi y,y}) + 2\mu U_{\xi x,x}$$

$$\sigma_{x\eta} = \lambda (U_{\eta x,x} + U_{\eta y,y}) + 2\mu U_{\eta x,x}$$

$$\left. \right\}$$

$$(7.92)$$

$$\left. \overline{\sigma}_{x\xi} = \lambda \left(T_{\xi x,x} + T_{\xi y,y} \right) + 2\mu T_{\xi x,x} \\ \overline{\sigma}_{x\eta} = \lambda \left(T_{\eta x,x} + T_{\eta y,y} \right) + 2\mu T_{\eta x,x}$$

$$\left. \right\}$$

$$(7.93)$$

Eq. (7.91) becomes

$$\sigma_{x} = \int_{\Omega} \left(\sigma_{x\xi} b_{\xi} + \sigma_{x\eta} b_{\eta} \right) d\Omega + \int_{\Gamma} \left(\sigma_{x\xi} t_{\xi} + \sigma_{x\eta} t_{\eta} \right) ds$$
$$- \int_{\Gamma} \left(\overline{\sigma}_{x\xi} u + \overline{\sigma}_{x\eta} v \right) ds \tag{7.94}$$

The quantities $\sigma_{x\xi}$ and $\sigma_{x\eta}$ express the stresses σ_x at point $(x, y) \in \Omega$ for unit loads applied at point (ξ, η) in the x- and y-direction, respectively (see Eqs. (7.65) and (7.68)). Similarly, the quantities $\overline{\sigma}_{x\xi}$ and $\overline{\sigma}_{x\eta}$ express the stresses σ_x at point $(x, y) \in \Omega$, due to unit displacements applied at point (ξ, η) in the xand y-direction, respectively.

Similarly, we obtain

$$\sigma_{y} = \int_{\Omega} \left(\sigma_{y\xi} b_{\xi} + \sigma_{y\eta} b_{\eta} \right) d\Omega + \int_{\Gamma} \left(\sigma_{y\xi} t_{\xi} + \sigma_{y\eta} t_{\eta} \right) ds$$
$$- \int_{\Gamma} \left(\overline{\sigma}_{y\xi} u + \overline{\sigma}_{y\eta} v \right) ds$$
(7.95)

$$\tau_{xy} = \int_{\Omega} \left(\tau_{xy\xi} b_{\xi} + \tau_{xy\eta} b_{\eta} \right) d\Omega + \int_{\Gamma} \left(\tau_{xy\xi} t_{\xi} + \tau_{xy\eta} t_{\eta} \right) ds$$
$$- \int_{\Gamma} \left(\overline{\tau}_{xy\xi} u + \overline{\tau}_{xy\eta} v \right) ds \tag{7.96}$$

where

$$\sigma_{y\xi} = \lambda (U_{\xi x,x} + U_{\xi y,y}) + 2\mu U_{\xi y,y} \sigma_{y\eta} = \lambda (U_{\eta x,x} + U_{\eta y,y}) + 2\mu U_{\eta y,y}$$
(7.97)

$$\left. \overline{\sigma}_{y\xi} = \lambda \left(T_{\xi x,x} + T_{\xi y,y} \right) + 2\mu T_{\xi y,y} \\ \overline{\sigma}_{y\eta} = \lambda \left(T_{\eta x,x} + T_{\eta y,y} \right) + 2\mu T_{\eta y,y} \right\}$$

$$(7.98)$$

$$\tau_{xy\xi} = \mu(U_{\xi x,y} + U_{\xi y,x})$$

$$\tau_{xy\eta} = \mu(U_{\eta x,y} + U_{\eta y,x})$$

$$(7.99)$$

$$\left. \begin{aligned} \overline{\tau}_{xy\xi} &= \mu \left(T_{\xi x,y} + T_{\xi y,x} \right) \\ \overline{\tau}_{xy\eta} &= \mu \left(T_{\eta x,y} + T_{\eta y,x} \right) \end{aligned} \right\}$$
(7.100)

The stresses $\sigma_{x\xi}$, $\sigma_{x\eta}$, $\sigma_{y\xi}$, $\sigma_{y\eta}$, $\tau_{xy\xi}$ and $\tau_{xy\eta}$ are given by Eqs. (7.66) and (7.69), whereas $\overline{\sigma}_{x\xi}$, $\overline{\sigma}_{x\eta}$, $\overline{\sigma}_{y\xi}$, $\overline{\sigma}_{y\eta}$, $\overline{\tau}_{xy\xi}$ and $\overline{\tau}_{xy\eta}$ are derived by introducing

Eqs. (7.71) and (7.72) in Eqs. (7.93), (7.98) and (7.100), and by performing the involved differentiations. Thus, we obtain

$$\overline{\sigma}_{x\xi} = \frac{A_3}{r^2} \left[2 r_{,x} r_{,n} \left(1 - 4 r_{,x}^2 \right) + \left(2 r_{,x}^2 + 1 \right) n_x \right]
\overline{\sigma}_{x\eta} = \frac{A_3}{r^2} \left(-8 r_{,x}^2 r_{,y} r_{,n} + 2 r_{,x} r_{,y} n_x + n_y \right)
\overline{\sigma}_{y\xi} = \frac{A_3}{r^2} \left(-8 r_{,y}^2 r_{,x} r_{,n} + 2 r_{,x} r_{,y} n_y + n_x \right)
\overline{\sigma}_{y\eta} = \frac{A_3}{r^2} \left[2 r_{,y} r_{,n} \left(1 - 4 r_{,y}^2 \right) + \left(2 r_{,y}^2 + 1 \right) n_y \right]
\overline{\tau}_{xy\xi} = \overline{\sigma}_{x\eta}
\overline{\tau}_{xy\eta} = \overline{\sigma}_{y\xi}$$
(7.101)

where $A_3 = -2\mu A_1$.

7.10 Numerical solution of the boundary integral equations

7.10.1 Evaluation of the unknown boundary quantities

The boundary integral equations are solved using the BEM with constant boundary elements. The boundary is divided into N constant elements. Thus the distribution of the displacements and tractions are taken constant on each element and equal to their value at the nodal point, which lies at the midpoint of the element.

Denoting by $\{u\}^i = \{u^i \ v^i\}^T$ and $\{t\}^i = \{t^i_x \ t^i_y\}^T$ the displacements and tractions at the *i*-th node and taking into account that the boundary is smooth at the nodal point of the constant element, Eqs. (7.89) can be written as

$$\frac{1}{2} \{u\}^{i} + \sum_{j=1}^{N} [\hat{H}]^{ij} \{u\}^{j} = \sum_{j=1}^{N} [G]^{ij} \{t\}^{j} + \{F\}^{i}$$
(7.102)

where

$$[G]^{ij} = \begin{bmatrix} \int_{\Gamma_j} U_{\xi x}(q, p_i) ds_q & \int_{\Gamma_j} U_{\eta x}(q, p_i) ds_q \\ \int_{\Gamma_j} U_{\xi y}(q, p_i) ds_q & \int_{\Gamma_j} U_{\eta y}(q, p_i) ds_q \end{bmatrix}$$
(7.103)

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$$[\hat{H}]^{ij} = \begin{bmatrix} \int_{\Gamma_j}^{\Gamma} T_{\xi x}(q, p_i) ds_q & \int_{\Gamma_j}^{\Gamma} T_{\eta x}(q, p_i) ds_q \\ \\ \int_{\Gamma_j}^{\Gamma} T_{\xi y}(q, p_i) ds_q & \int_{\Gamma_j}^{\Gamma} T_{\eta y}(q, p_i) ds_q \end{bmatrix}$$
(7.104)

and

$$\{F\}^{i} = \begin{cases} \int_{\Omega} \left[U_{\xi x}(Q, p_{i}) b_{\xi}(Q) + U_{\eta x}(Q, p_{i}) b_{\eta}(Q) \right] d\Omega_{Q} \\ \int_{\Omega} \left[U_{\xi y}(Q, p_{i}) b_{\xi}(Q) + U_{\eta y}(Q, p_{i}) b_{\eta}(Q) \right] d\Omega_{Q} \end{cases}$$
(7.105)

with $p_i, q \in \Gamma$ and $Q \in \Omega$.

Equation (7.102) relates the displacements of the i-th node to the displacements and tractions of all the nodes including the i-th node.

Applying Eq. (7.102) to all the boundary nodal points yields 2N equations, which can be set in matrix form as

$$[H]{u} = [G]{t} + {F}$$
(7.106)

where

$$[H] = [\hat{H}] + \frac{1}{2}[I] \tag{7.107}$$

The dimensions of the matrices $[\hat{H}]$ and [G] are $2N \times 2N$, and those of the vectors $\{u\}$, $\{t\}$ and $\{F\}$ are 2N. They are defined as

$$[G] = \begin{bmatrix} [G]^{11} & [G]^{12} & \cdots & [G]^{1N} \\ [G]^{21} & [G]^{22} & \cdots & [G]^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ [G]^{N1} & [G]^{N2} & \cdots & [G]^{NN} \end{bmatrix}$$
(7.108)
$$[\hat{H}] = \begin{bmatrix} [\hat{H}]^{11} & [\hat{H}]^{12} & \cdots & [\hat{H}]^{1N} \\ [\hat{H}]^{21} & [\hat{H}]^{22} & \cdots & [\hat{H}]^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ [\hat{H}]^{N1} & [\hat{H}]^{N2} & \cdots & [\hat{H}]^{NN} \end{bmatrix}$$
(7.109)

and

$$\{u\} = \begin{cases} \{u\}^{1} \\ \{u\}^{2} \\ \vdots \\ \{u\}^{N} \end{cases}, \quad \{t\} = \begin{cases} \{t\}^{1} \\ \{t\}^{2} \\ \vdots \\ \{t\}^{N} \end{cases}, \quad \{F\} = \begin{cases} \{F\}^{1} \\ \{F\}^{2} \\ \vdots \\ \{F\}^{N} \end{cases}$$
(7.110)

The 2N equations within the matrix Eq. (7.106) contain 4N boundary values, that is 2N values of displacements and 2N values of tractions. However, a total of 2N values are known from the boundary conditions. Consequently, Eqs. (7.106) can be used to determine the 2N unknown boundary values. It should be noted that rearrangement of the unknowns is necessary for mixed boundary conditions. After doing so the following system of 2N linear equations is obtained

$$[A]{X} = {R} + {F}$$
(7.111)

where [A] is a square coefficient matrix having dimensions $2N \times 2N$, $\{X\}$ is the vector including the 2N unknown boundary values and $\{R\}$ is a vector resulting as the sum of the columns of the matrices [G] and [H] multiplied by the respective known boundary values. Columns originating from matrices [H] and [G] should have their sign switched, when they are moved to the other side of the equation. Special attention should be paid in the case where only boundary tractions are prescribed, that is when the vector $\{t\}$ in Eq. (7.106) is known. For boundary conditions of this kind, (case (iv) of Eqs. (7.21), the displacements are not determined uniquely, because they also include a rigid body motion. This is reflected in the matrix [H] whose rank is 2N - 3 and therefore can not be inverted. To overcome this problem we restrain the rigid body motion. For this purpose the body is supported by setting three elements of the vector $\{u\}$ equal to zero. Attention should be paid in order to exclude any infinitesimal kinematic indeterminacy of the body, which would lead to an ill-conditioned matrix [A].

7.10.2 Evaluation of displacements in the interior of the body

The system of Eqs. (7.111) is solved for the unknown boundary values of the displacements and tractions. Thereafter all the boundary quantities are known and, consequently, the displacements at any point $P_i(x_i, y_i)$ inside the domain Ω can be evaluated using Eq. (7.76), which after discretization becomes

$$\{u\}^{i} = \sum_{j=1}^{N} [G]^{ij} \{t\}^{j} - \sum_{j=1}^{N} [\hat{H}]^{ij} \{u\}^{j} + \{F\}^{i}$$
(7.112)

The matrices $[G]^{ij}$ and $[\hat{H}]^{ij}$ as well as the vector $\{F\}^i$ are evaluated according to Eqs. (7.103), (7.104) and (7.105), respectively, with $P \in \Omega$ in place of $p_i \in \Gamma$. The superscript *i* is pertaining now to point $P_i(x_i, y_i)$ inside Ω and not to one of the boundary nodes.

7.10.3 Evaluation of stresses in the interior of the body

The stresses at any point $P_i(x_i, y_i)$ inside the domain Ω are evaluated using Eqs. (7.94), (7.95) and (7.96), which after discretization are written as

$$\begin{cases} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{cases}^i = \sum_{j=1}^N [\sigma]^{ij} \{t\}^j - \sum_{j=1}^N [\overline{\sigma}]^{ij} \{u\}^j + \{S\}^i$$
(7.113)

The matrices $[\sigma]^{ij}$ and $[\overline{\sigma}]^{ij}$ as well as the vector $\{S\}^i$ are computed from the following relations

$$[\sigma]^{ij} = \begin{bmatrix} \int_{\Gamma_j} \sigma_{x\xi}(q, P_i) ds_q & \int_{\Gamma_j} \sigma_{x\eta}(q, P_i) ds_q \\ \int_{\Gamma_j} \sigma_{y\xi}(q, P_i) ds_q & \int_{\Gamma_j} \sigma_{y\eta}(q, P_i) ds_q \\ \int_{\Gamma_j} \tau_{xy\xi}(q, P_i) ds_q & \int_{\Gamma_j} \tau_{xy\eta}(q, P_i) ds_q \end{bmatrix}$$
(7.114)

$$\{\overline{\sigma}\}^{ij} = \begin{bmatrix} \int_{\Gamma_j} \overline{\sigma}_{x\xi}(q, P_i) ds_q & \int_{\Gamma_j} \overline{\sigma}_{x\eta}(q, P_i) ds_q \\ \int_{\Gamma_j} \overline{\sigma}_{y\xi}(q, P_i) ds_q & \int_{\Gamma_j} \overline{\sigma}_{y\eta}(q, P_i) ds_q \\ \int_{\Gamma_j} \overline{\tau}_{xy\xi}(q, P_i) ds_q & \int_{\Gamma_j} \overline{\tau}_{xy\eta}(q, P_i) ds_q \end{bmatrix}$$
(7.115)

$$\{S\}^{i} = \begin{cases} \int_{\Omega} \left[\sigma_{\xi}(Q, P_{i}) b_{\xi}(Q) + \sigma_{\eta}(Q, P_{i}) b_{\eta}(Q) \right] d\Omega_{Q} \\ \int_{\Omega} \left[\sigma_{\xi}(Q, P_{i}) b_{\xi}(Q) + \sigma_{\eta}(Q, P_{i}) b_{\eta}(Q) \right] d\Omega_{Q} \\ \int_{\Omega} \left[\tau_{xy\xi}(Q, P_{i}) b_{\xi}(Q) + \tau_{yx\eta}(Q, P_{i}) b_{\eta}(Q) \right] d\Omega_{Q} \end{cases}$$
(7.116)

7.10.4 Evaluation of stresses on the boundary

The stresses σ_x , σ_y and τ_{xy} on the boundary can be evaluated from Eqs. (7.94), (7.95) and (7.96) by letting point $P \in \Omega$ approach point $p \in \Gamma$ and following a limiting process similar to that presented in Section 7.8. This, however, is not recommended due to the difficulties arising from the behavior of the line integrals as $P \in \Omega$ moves to $p \in \Gamma$, and on the other hand from the need to treat singular and hyper-singular line integrals. For this reason, the technique that follows is preferred as a much simpler and straight forward one. The derivatives of the displacements u and v along the tangent to the boundary are given as

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}n_y + \frac{\partial u}{\partial y}n_x$$

$$\frac{\partial v}{\partial t} = -\frac{\partial v}{\partial x}n_y + \frac{\partial v}{\partial y}n_x$$
(7.117)

These derivatives can be computed from the boundary values of u and v through numerical differentiation (see Section 6.2.2).

Collecting Eqs. (7.23) and (7.117), we can write

$$\begin{vmatrix} (\lambda + 2\mu) n_x & \mu n_y & \mu n_y & \lambda n_x \\ \lambda n_y & \mu n_x & \mu n_x & (\lambda + 2\mu) n_y \\ -n_y & n_x & 0 & 0 \\ 0 & 0 & -n_y & n_x \end{vmatrix} \begin{cases} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{cases} = \begin{cases} t_x \\ t_y \\ u_{,t} \\ v_{,t} \end{cases}$$
(7.118)

Taking into account that $n_x^2 + n_y^2 = 1$, it can be readily shown that the determinant of the coefficient matrix in Eq. (7.118) is

$$D = -\mu \left(\lambda + 2\mu\right) \neq 0 \tag{7.119}$$

Hence the derivatives of u and v with respect to x and y can always be evaluated and the stresses can be computed from the expressions

$$\sigma_{x} = \lambda (u_{,x} + v_{,y}) + 2\mu u_{,x}$$

$$\sigma_{y} = \lambda (u_{,x} + v_{,y}) + 2\mu u_{,y}$$

$$\tau_{xy} = \mu (u_{,y} + v_{,x})$$

$$(7.120)$$

7.11 Body forces

It becomes apparent from Eqs. (7.105) and (7.116) that the inclusion of body forces requires the evaluation of domain integrals. These integrals can be treated in one of the following ways.

7.11.1 Direct numerical evaluation

The domain Ω is discretized into cells and the integration is performed using the procedure developed for Poisson's equation in Section 4.4. This method, however, has two drawbacks. The first one is that it diminishes the elegance and computa-

tional efficiency of the BEM by involving the difficult problem of domain discretization. Of course, the problem does not appear as complex as in the FEM, where the continuous body is approximated by the discretized one. In the BEM, the discretization is employed only to approximate the integral by a sum. Even the assumption of constant value of the integrand on each cell (discontinuous finite element) results in a good approximation for the integral. This fact is a consequence of the mean value theorem of integral calculus.

The second drawback of the domain integration originates from Eqs. (7.58), (7.60), (7.66), (7.69) and (7.101) whose integrands behave like lnr, 1/r, $1/r^2$ or $1/r^3$, where r = |P - Q|. These integrals become singular or hyper-singular, when point P lies on the cell over which the integration is performed, because point Q also lies on the same cell and thus r will take among others the value r = 0. Several methods have been developed for the evaluation of these integrals [7]. A relatively simple but effective and accurate method is that developed by Katsikadelis [8], which converts the singular or hyper-singular domain integrals to regular line integrals on the boundary of the cell. This method is presented in Appendix B.

7.11.2 Evaluation using a particular solution

In this case the solution of Navier's equations, Eqs. (7.19), is obtained as a sum of two solutions

$$u = u_0 + u_1, \quad v = v_0 + v_1$$
 (7.121)

where u_0 , v_0 is the solution of the homogeneous equation and u_1 , v_1 a particular one of the non-homogeneous equation. Using the notation of Eqs. (7.45) for the differential operators, the Navier equations are written as

$$\begin{cases} N_x(u,v) = b_x \\ N_y(u,v) = b_y \end{cases}$$
 in Ω (7.122)

or using Eqs. (7.121) and taking into account that the operators N_x and N_y are linear, we can obtain the solution u_0 , v_0 from the following boundary value problem

$$\begin{cases} N_x(u_0, v_0) = 0 \\ N_y(u_0, v_0) = 0 \end{cases}$$
 in Ω (7.123)

and

- (i) $u_0 = \overline{u} u_1$, $v_0 = \overline{v} v_1$ on Γ_1 (7.124a)
- (ii) $u_0 = \overline{u} u_1$, $(t_y)_0 = \overline{t}_y t_y^1$ on Γ_2 (7.124b)

- (iii) $(t_x)_0 = \overline{t_x} t_x^1$, $v_0 = \overline{v} v_1$ on Γ_3 (7.124c)
- (iv) $(t_x)_0 = \overline{t_x} t_x^1$, $(t_y)_0 = \overline{t_y} t_y^1$ on Γ_4 (7.124d)

The above types of boundary conditions result from Eqs. (7.21) when these are expressed in terms of u_0 and v_0 . The traction components t_x^1 and t_y^1 are obtained from Eqs. (7.23), if u, v are replaced by u_1 , v_1 .

The particular solution is obtained from the equations

$$\left.\begin{array}{l} N_x(u_1,v_1)=b_x\\ \\ N_y(u_1,v_1)=b_y\end{array}\right\}$$

It is apparent that the determination of the particular solution should precede the solution of the homogeneous equations. The particular solution can be obtained from Eqs. (7.49)

$$2G u_{1} = \frac{2}{1+\overline{\nu}} \nabla^{2} \phi_{1} - \frac{\partial}{\partial x} \left(\frac{\partial \phi_{1}}{\partial x} + \frac{\partial \psi_{1}}{\partial y} \right)$$

$$2G v_{1} = \frac{2}{1+\overline{\nu}} \nabla^{2} \psi_{1} - \frac{\partial}{\partial y} \left(\frac{\partial \phi_{1}}{\partial x} + \frac{\partial \psi_{1}}{\partial y} \right)$$

$$(7.125)$$

The functions ϕ_1 , ψ_1 are the components of the Galerkin vector and are determined as a particular solution of the following equations

$$\nabla^4 \phi_1 = -(1+\bar{\nu})b_x \tag{7.126a}$$

$$\nabla^4 \psi_1 = -(1+\overline{\nu}) b_y \tag{7.126b}$$

Equations (7.126a) and (7.126b) represent the bending equation of a thin plate subjected to transverse loads $-(1 + \overline{\nu})b_x$ and $-(1 + \overline{\nu})b_y$, respectively. A particular solution of these equations can be obtained using the method presented in [9]. This method is an extension of that presented in Section 3.4.2 for the harmonic equation to the case of the biharmonic equation.

By introducing the variables

z = x + iy and $\overline{z} = x - iy$ (7.127)

Eq. (7.126a) is transformed to

$$16 \frac{\partial^4 \overline{\phi}_1}{\partial z^2 \partial \overline{z}^2} = b_x(z, \overline{z})$$
(7.128)

where it has been set

$$\overline{\phi}_1 = -\frac{\phi_1}{(1+\overline{\nu})} \tag{7.129}$$

A particular solution $\overline{\phi}_1(z,\overline{z})$ is readily obtained by integrating Eq. (7.128) consecutively four times. The resulting arbitrary integration functions can be neglected, since we are seeking a particular solution.

Subsequently, back substitution of z and \overline{z} by virtue of Eqs. (7.127) yields $\overline{\phi}_1(x,y)$. Similarly, we find the particular solution $\overline{\psi}_1(x,y)$ of Eq. (7.126b).

Example 7.1

Determine the particular solution, if the body force $b_x = \rho g \cos \theta$, $b_y = \rho g \sin \theta$ is the weight of the body, which acts in the direction of the vector $(\cos \theta, \sin \theta)$. We denote by ρ the mass density per unit area and by g the acceleration of gravity.

First we establish the components of the Galerkin vector. Equation (7.128) is written

$$16 \frac{\partial^4 \overline{\phi}_1}{\partial z^2 \partial \overline{z}^2} = \rho g \cos \theta$$

Consecutive integrations yield

$$\overline{\phi}_1 = \frac{\rho g \cos \theta}{64} z^2 \overline{z}^2 = \frac{\rho g}{64} r^4 \cos \theta$$

or

$$\phi_1 = k r^4 \cos heta$$

where it was set

$$k = -\frac{\rho g (1 + \overline{\nu})}{64}$$
 and $r^2 = x^2 + y^2$

Similarly, we have

$$\psi_1 = k r^4 \sin \theta$$

Substituting the above expressions for ϕ_1 and ψ_1 into Eqs. (7.125) yields

$$u_{1} = -\frac{\rho g}{4G} \left\{ r^{2} \cos \theta - \frac{1+\overline{\nu}}{8} \left[\left(3x^{2} + y^{2} \right) \cos \theta + 2xy \sin \theta \right] \right\}$$

and
$$v_1 = -\frac{\rho g}{4G} \left\{ r^2 \sin \theta - \frac{1+\overline{\nu}}{8} \left[2xy \cos \theta + \left(x^2 + 3y^2 \right) \sin \theta \right] \right\}$$

If the y-axis is taken in the vertical direction, it will be $\theta = -\pi/2$, $\cos \theta = 0$, $\sin \theta = -1$, and the particular solution is simplified as

$$u_{1} = -\frac{\rho g (1 + \overline{\nu})}{16G} xy$$
$$v_{1} = -\frac{\rho g}{16G} \left[-4 \left(x^{2} + y^{2} \right) + \frac{1}{2} (1 + \overline{\nu}) \left(x^{2} + 3y^{2} \right) \right]$$

7.11.3 Transformation of the domain integrals to boundary integrals

This method is closer to BEM's logic, since domain integration is avoided and the pure boundary character of the method is maintained. The transformation of the domain integrals to boundary line integrals may be accomplished by different techniques. Two methods are going to be presented in the sequel. One is general and it applies to body forces having an arbitrary distribution, while the other one is used for body forces which are derived from a potential function.

(a) Body forces having arbitrary distribution

This method is analogous to that presented for Poisson's equation in Section 3.5 (ii) [10].

First, using the procedure presented in the previous section, we determine a particular solution u_1 , v_1 for Navier's equations,

$$N_x(u_1,v_1)=b_x$$

$$N_y(u_1,v_1)=b_y$$

Next the reciprocal relation (7.43) is employed consecutively for

(i) $u = u_1, v = v_1$ and $u^* = U_{\xi x}, v^* = U_{\eta x}$

(ii)
$$u = u_1, v = v_1$$
 and $u^* = U_{\xi_y}, v^* = U_{\eta_y}$

and having in mind that the fundamental solution $U_{\xi x}$, $U_{\eta x}$, $U_{\xi y}$, $U_{\eta y}$ satisfies the equations

$$N_{x}(U_{\xi x}, U_{\eta x}) = \delta(Q - P)$$

$$N_{y}(U_{\xi x}, U_{\eta x}) = 0$$

$$(7.130a)$$

and

$$\left. \begin{array}{l} N_x(U_{\xi y}, U_{\eta y}) = 0 \\ \\ N_y(U_{\xi y}, U_{\eta y}) = \delta(Q - P) \end{array} \right\}$$

$$(7.130b)$$

we obtain the components F_x and F_y of the vector (7.105) as

$$F_{x}(P) = \varepsilon_{11} u_{1}(P) + \varepsilon_{21} v_{1}(P) -\int_{\Gamma} \Big[U_{\xi x}(q, P) t_{\xi}^{1}(q) + U_{\eta x}(q, P) t_{\eta}^{1}(q) \Big] ds_{q} +\int_{\Gamma} \Big[T_{\xi x}(q, P) u_{1}(q) + T_{\eta x}(q, P) v_{1}(q) \Big] ds_{q}$$
(7.131)

$$F_{y}(P) = \varepsilon_{12} u_{1}(P) + \varepsilon_{22} v_{1}(P)$$

- $\int_{\Gamma} \Big[U_{\xi y}(q, P) t_{\xi}^{1}(q) + U_{\eta y}(q, P) t_{\eta}^{1}(q) \Big] ds_{q}$
+ $\int_{\Gamma} \Big[T_{\xi y}(q, P) u_{1}(q) + T_{\eta y}(q, P) v_{1}(q) \Big] ds_{q}$ (7.132)

where the coefficients ε_{ij} take the value $\varepsilon_{ij} = \delta_{ij}$ for $P \in \Omega$, while for $P \in \Gamma$ they are evaluated from Eqs. (7.84)–(7.87). If the boundary is smooth at point $P \in \Gamma$, it is $\varepsilon_{ij} = \frac{1}{2}\delta_{ij}$.

It should be noted that Eqs. (7.131) and (7.132) hold for body forces b_{ξ} and b_{η} which are distributed over the whole domain Ω . If b_{ξ} or b_{η} act only on a subregion $\Omega^* \subset \Omega$, then for points P outside Ω^* it is $\varepsilon_{ij} = 0$ and thus terms outside the integrals vanish. Therefore, Eqs. (7.131) and (7.132) can only be employed for points $P \in (\Omega^* \cup \Gamma^*)$.

(b) Body forces derived from a potential function

The previous case, where b_{ξ} and b_{η} are arbitrary functions, is rather theoretical. On the other hand, the case where the body forces are derived from a potential function is important and of special practical interest, e.g. gravitational forces. The conversion of the domain integrals to boundary integrals can be accomplished as follows.

If $V = V(\xi, \eta)$ is the function that represents the potential, then the components of the body force are obtained as

$$b_{\xi} = \frac{\partial V}{\partial \xi}, \quad b_{\eta} = \frac{\partial V}{\partial \eta}$$
 (7.133)

Moreover, the potential function satisfies the equation

$$\nabla^2 V = 0 \tag{7.134}$$

The integral F_x defined by Eq. (7.105) is written as

$$F_{x} = \int_{\Omega} \left(U_{\xi x} V_{\xi} + U_{\eta x} V_{\eta} \right) d\Omega$$

= $-\int_{\Omega} \left(U_{\xi x,\xi} + U_{\eta x,\eta} \right) V d\Omega + \int_{\Gamma} \left(U_{\xi x} n_{x} + U_{\eta x} n_{y} \right) V ds$ (7.135)

where n_x and n_y are the direction cosines of the unit vector **n** which is normal to the boundary at point (ξ, η) . The above transformation of the domain integral is achieved by employing Eqs. (2.7) and (2.8).

Incorporating Eqs. (7.58), the domain integral of the right-hand side of Eq. (7.135) becomes

$$\int_{\Omega} \left(U_{\xi x,\xi} + U_{\eta x,\eta} \right) V \, d\Omega = -\frac{2(1-\overline{\nu})}{8\pi G} \int_{\Omega} (\ell n r)_{\xi} \, V \, d\Omega \tag{7.136}$$

Further, setting

$$\phi = \frac{1}{4}r^2\ell nr \tag{7.137}$$

we find that

$$abla^2 \phi_{,\xi} = (\ell n r)_{,\xi}$$

and then applying Green's identity (2.16) for v = V and $u = \phi_{\xi}$ we obtain

$$\int_{\Omega} V \nabla^2 \phi_{\xi} \, d\Omega = \int_{\Gamma} \left(V \frac{\partial \phi_{\xi}}{\partial n} - \phi_{\xi} \, \frac{\partial V}{\partial n} \right) ds \tag{7.138}$$

Hence, Eq. (7.135) can be finally written as

$$F_{x} = \frac{(1-\vec{\nu})}{4\pi G} \int_{\Gamma} \left(V \frac{\partial \phi_{\xi}}{\partial n} - \phi_{\xi} \frac{\partial V}{\partial n} \right) ds + \int_{\Gamma} \left(U_{\xi x} n_{x} + U_{\eta x} n_{y} \right) ds$$
(7.139)

Similarly, the contribution of the body forces in the y-direction may be obtained in terms of boundary-only integrals as

$$F_{y} = \frac{(1-\overline{\nu})}{4\pi G} \int_{\Gamma} \left(V \frac{\partial \phi_{,\eta}}{\partial n} - \phi_{,\eta} \frac{\partial V}{\partial n} \right) ds + \int_{\Gamma} \left(U_{\xi y} n_{x} + U_{\eta y} n_{y} \right) ds$$
(7.140)

7.12 Program ELBECON for solving the plane elastostatic problem with constant boundary elements

On the basis of the analysis presented in the previous sections a computer program has been written in the FORTRAN language [6]. It solves the two plane elasticity problems, namely the plane strain and the plane stress for plane bodies which may have holes. For the simplicity of the program, the body forces are not included. The program employs constant elements for the discretization of the boundary integral equations. The structure of program ELBECON is shown in the macro flow chart of Fig. 7.7.

Main program

The main program defines the parameters N, NB and IN, which specify the number of boundary elements, the number of boundaries and the number of internal points where the solution will be computed, respectively. It opens two files, the file INPUTFILE, which contains the data and the OUTPUTFILE file, in which the results are rendered. Subsequently it calls the following ten subroutines:

INPUTEL	Reads the data from INPUTFILE.
GMATREL	Forms the matrix $[G]$ defined by Eq. (7.108).
HMATREL	Forms the matrix $[H]$ defined by Eqs. (7.107) and (7.109).
ABMATREL	Rearranges the matrices $[H]$ and $[G]$ on the basis of the given boundary conditions and forms the matrix $[A]$ and the vector $\{B\} = \{R\}$ of Eq. (7.111).
SOLVEQ	Solves the system of linear equations $[A]{X} = {R}$ using Gauss elimination.
REORDEREL	Rearranges the boundary values and forms the matrices $\{u\}$, $\{v\}$, $\{t_x\}$ and $\{t_y\}$.
UVINTER	Computes the displacements u and v at the internal points using Eq. (7.112).
STRESSB	Computes the stresses σ_x , σ_y and τ_{xy} at the boundary nodal points using Eqs. (7.120).
STRESSIN	Computes the stresses σ_x , σ_y and τ_{xy} at the internal points using Eqs. (7.113).
OUTPUTEL	Writes the results in OUTPUTFILE.

The variables and the arrays used in the program together with their meaning are given below:

N Total number of boundary elements and hence of boundary nodes for constant boundary elements.



Figure 7.7 Macro flow chart of program ELBECON.

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IN	Number of internal points, where the displacements and the stresses are computed.
NB	Number of boundaries.
NL	One-dimensional array of dimension NB containing the serial number of the last element of each boundary.
IPLANE	Integer specifying the type of elasticity problem. IPLANE=0 for plane strain; IPLANE=1 for plane stress.
EL	Young's modulus of elasticity, E .
GL	Shear modulus, G.
PN	Poisson's ratio, ν .
KCODE	One-dimensional array specifying the type of boundary condi- tions at node I (I=1,2,,N) and taking the values: KCODE(I)=1 when u and v are prescribed KCODE(I)=2 when t_x and t_y are prescribed KCODE(I)=3 when u and t_y are prescribed KCODE(I)=4 when t_x and v are prescribed
XL, YL	One-dimensional arrays containing the x and y coordinates of the extreme points of the elements.
XM, YM	One-dimensional arrays containing the x and y coordinates of the boundary nodes.
XIN, YIN	One-dimensional arrays containing the x and y coordinates of the internal points, where the displacements and stresses are computed.
R	One-dimensional array defined in Eq. (7.111).
UB, VB	One-dimensional arrays. At input they contain the prescribed boundary values, i.e. displacements and/or tractions. At output they contain the boundary nodal values of u and v .
TXB, TYB	One-dimensional arrays containing at output the values of the boundary tractions t_x and t_y .
UIN, VIN	One-dimensional arrays containing the computed values of the displacements u and v at the internal points.
SXIN, SYIN, SXYIN	One-dimensional arrays containing the computed values of the stress components σ_x , σ_y and τ_{xy} at the IN internal points.
SXB, SYB, SXYB	One-dimensional arrays containing the computed values of the stress components σ_x , σ_y and τ_{xy} at the N boundary points.

Subroutine INPUTEL

The subroutine INPUTEL reads all the data required by the program using free FORMAT. The data have been written in INPUTFILE, to which the user has given a specific name as required by the main the program. This file contains the following data:

- 1. User's name. One line containing the name of the user.
- 2. Title. One line containing the title of the program.
- 3. *The code number of the problem type*. IPLANE=0 for plane strain; IPLANE=1 for plane stress.
- 4. Elastic constants. Modulus of elasticity EL and Poisson's ratio PN.
- 5. *The number of the last element of each boundary.* NB integers forming the entries of the array NL.
- 6. *The extreme points of the boundary elements.* N couples of values consisting of the coordinates XL, YL of the extreme points of the boundary elements. They are read in the positive sense, that is, counter-clockwise on the external boundary, and clockwise on the internal ones.
- 7. Boundary conditions. N triples of numbers consisting of the values of KCODE, u or t_x and v or t_y . More specifically:

8. Coordinates of the internal points. IN couples of values consisting of the coordinates XIN and YIN of the internal points, where the values of the displacements u, v and stresses σ_x , σ_y , τ_{xy} will be computed.

Finally, the subroutine INPUTEL writes the data in OUTPUTFILE, to which the user gives a specific name.

Subroutine GMATREL

The subroutine GMATREL forms the matrix [G] defined by Eq. (7.108). First, the submatrices $[G]^{ij}$ (i, j = 1, 2, ..., N), given by Eq. (7.103) are evaluated and subsequently placed in matrix [G]. The elements of the matrix $[G]^{ij}$ are line integrals of the fundamental solution along the constant boundary element. We distinguish two cases for the position of the elements with respect to the diagonal of the matrix:

(i) Off-diagonal elements, $i \neq j$

In this case the reference point $P_i(x_i, y_i)$ lies outside the *j*-th element, over which the integration is performed, and thus the distance $r = |q - P_i|$ never vanishes. Consequently, the line integrals in Eq. (7.103) are regular and they are evaluated

using a four-point Gauss integration by calling the subroutine RLINTG (see program LABECON).

(ii) Diagonal elements, i = j

In this case the reference point $P_i(x_i, y_i)$ lies on the j-th element, over which the integration is performed. The distance $r = |q - P_i|$ vanishes for $P_i \equiv q$ and consequently the line integrals in Eq. (7.103) become singular. For this reason, their evaluation is carried out analytically in the interval $[\varepsilon, \ell_i/2]$ and then, we take the limit $\varepsilon \to 0$. It should be mentioned that the integral in the interval $[\varepsilon, \ell_i/2]$ is equal to that in $[-\ell_i/2, -\varepsilon]$.



Figure 7.8 Element subdivision for the analytical evaluation of the singular integrals of submatrices $[G]^n$.

Noting that

$$r_{x} = \cos \alpha = \frac{x_{i+1} - x_{i}}{\ell_{i}}, \quad r_{y} = \sin \alpha = \frac{y_{i+1} - y_{i}}{\ell_{i}}$$
 (7.141)

where $\alpha = \text{angle}(\mathbf{x}, \mathbf{r}) = \text{constant}$ for each element (see Fig. 7.8), Eqs. (7.58), (7.60) and (7.103) yield

$$G_{11}^{ii} = \int_{\Gamma_{i}} U_{\xi x} \, ds = \int_{\Gamma_{i}} U_{x\xi} \, ds$$

$$= -\frac{1}{8\pi G} \lim_{\epsilon \to 0} \left\{ 2(3-\overline{\nu}) \int_{\epsilon}^{\ell_{i}/2} \ell nr \, dr - 2(1+\overline{\nu}) \cos^{2} \alpha \int_{\epsilon}^{\ell_{i}/2} dr \right\}$$

$$= -\frac{1}{8\pi G} \lim_{\epsilon \to 0} \left\{ 2(3-\overline{\nu}) \left[r \, \ell nr - r \right]_{\epsilon}^{\ell_{i}/2} - 2(1+\overline{\nu}) \cos^{2} \alpha \left[r \right]_{\epsilon}^{\ell_{i}/2} \right\}$$

$$= -\frac{\ell_{i}}{8\pi G} \left[(3-\overline{\nu}) \left(\ell n \frac{\ell_{i}}{2} - 1 \right) - (1+\overline{\nu}) \cos^{2} \alpha \right]$$
(7.142a)

$$G_{12}^{ii} = G_{21}^{ii} = \int_{\Gamma_i} U_{\xi y} \, ds = \int_{\Gamma_i} U_{y\xi} \, ds$$

$$= \frac{1}{8\pi G} \, \lim_{\epsilon \to 0} \left\{ (1 + \overline{\nu}) \, 2 \int_{\epsilon}^{\ell_i/2} \cos \alpha \, \sin \alpha \, dr \right\}$$

$$= \frac{1}{8\pi G} \, (1 + \overline{\nu}) \, 2 \cos \alpha \, \sin \alpha \int_{0}^{\ell_i/2} dr$$

$$= \frac{\ell_i}{8\pi G} \, (1 + \overline{\nu}) \cos \alpha \, \sin \alpha \qquad (7.142b)$$

$$G_{22}^{ii} = \int_{\Gamma_i} U_{\eta y} \, ds = \int_{\Gamma_i} U_{y\eta} \, ds$$

$$= -\frac{1}{8\pi G} \, \lim_{\epsilon \to 0} \left\{ 2(3 - \overline{\nu}) \int_{\epsilon}^{\ell_i/2} \ell nr \, dr - 2(1 + \overline{\nu}) \sin^2 \alpha \int_{\epsilon}^{\ell_i/2} dr \right\}$$

$$= -\frac{1}{8\pi G} \lim_{\varepsilon \to 0} \left\{ 2(3-\overline{\nu}) \left[r \,\ell nr - r \right]_{\varepsilon}^{\ell_i/2} - 2(1+\overline{\nu}) \sin^2 \alpha \left[r \right]_{\varepsilon}^{\ell_i/2} \right\}$$
$$= -\frac{\ell_i}{8\pi G} \left[(3-\overline{\nu}) \left(\ell n \frac{\ell_i}{2} - 1 \right) - (1+\overline{\nu}) \sin^2 \alpha \right]$$
(7.142c)

For the evaluation of G_{11}^{ij} and G_{22}^{ij} the constant term in the expressions of $U_{\xi x}$ and $U_{\eta y}$ was neglected, since it does not influence the integral equations. Indeed, the contribution of a constant C to Eqs. (7.76) is

$$C\left(\int_{\Omega} b_{\xi} d\Omega + \int_{\Gamma} t_{\xi} ds\right) = 0 \quad \text{in } u(p)$$

$$C\left(\int_{\Omega} b_{\eta} d\Omega + \int_{\Gamma} t_{\eta} ds\right) = 0 \quad \text{in } v(p)$$

$$(7.143)$$

according to the equilibrium equations of the external forces acting on the body in the x- and y-directions, respectively.

Subroutine HMATREL

This subroutine forms the matrix [H] defined by Eq. (7.107). First, it evaluates the submatrices $[\hat{H}]^{ij}$ (i, j = 1, 2, ..., N) using Eq. (7.104), which are then placed in matrix $[\hat{H}]$. We distinguish again two cases for the position of the elements with respect to the diagonal of the matrix:

(i) Off-diagonal elements, $i \neq j$

As it was explained in the case of matrices $[G]^{ij}$, the distance $r = |q - P_i|$ does not vanish. Therefore, the line integrals are always regular and can be evaluated using a four-point Gauss integration by calling subroutine RLINTH (see program LABECON).

(ii) Diagonal elements, i = j

In this case the reference point $P_i(x_i, y_i)$ lies on the *j*-th element over which the integration is performed. The distance $r = |q - P_i|$ vanishes for $P_i \equiv q$ and consequently the line integrals of Eq. (7.104) become singular.

The integration of the kernels given by Eqs. (7.71) and (7.72) requires evaluation of singular integrals having the form

$$\int_{\Gamma_i} \frac{r_{in}}{r} ds$$
 and $\int_{\Gamma_i} \frac{r_{it}}{r} ds$

which can also be written as (see Section 4.3)

$$\int_{\Gamma_t} \frac{\cos \phi}{r} \, ds$$
 and $\int_{\Gamma_t} \frac{\sin \phi}{r} \, ds$

where $\phi = \text{angle}(\mathbf{r}, \mathbf{n})$ is shown in Fig. 4.4.

The first integral equals to zero as it was proved in Eq. (4.28b). Moreover, using Eqs. (4.21) and (4.27), the second integral is written

$$\int_{\Gamma_{0}} \frac{\sin \phi}{r} ds = \int_{-1}^{1} \frac{\sin \phi}{|\xi|} d\xi$$
$$= -\int_{0}^{1} \frac{d|\xi|}{|\xi|} + \int_{0}^{-1} \frac{d|\xi|}{|\xi|} = 0$$

since $\phi = 3\pi/2$ when $0 \le \xi \le 1$ and $\phi = \pi/2$ when $-1 \le \xi \le 0$. Hence

$$[\hat{H}]^{ii} = [0]$$

and

$$[H]^{ii} = [\hat{H}]^{ii} + \frac{1}{2}[I] = \begin{bmatrix} 1/2 & 0\\ 0 & 1/2 \end{bmatrix}$$
(7.144)

Subroutine ABMATREL

This subroutine rearranges the columns of matrices [G] and [H], and creates the matrix [A] and the vector $\{R\}$ of Eq. (7.111). The columns of matrix [A] consist of all the columns of matrices [G] and [H] that correspond to the unknown boundary values of u, v, t_x and t_y . The vector $\{R\}$ results as the sum of those columns of [G] and [H] which correspond to the known values of u, v, t_x and t_y after they have been multiplied by the respective values. It should be noted that a change of sign occurs, when the columns of [G] or [H] are transferred to the other side of Eq. (7.106).

Subroutine SOLVEQ

This subroutine solves the system of equations AX = R calling the subroutine LEQS. The solution is obtained by Gauss elimination and stored in the vector R. The output parameter LSING takes the value LSING = 0, when the matrix A is regular, or LSING = 1, when the matrix A is singular (see program LABECON).

Subroutine REORDEREL

This subroutine rearranges the vector $\{R\}$ on the basis of the given boundary conditions and forms the vectors $\{u\}$, $\{v\}$, $\{t_x\}$ and $\{t_y\}$ of the boundary displacements and tractions.

Subroutine UVINTER

This subroutine computes the displacements u and v at internal points utilizing Eq. (7.112). The matrices $[G]^{ij}$ and $[\hat{H}]^{ij}$ are evaluated by subroutines RLINTG and RLINTH. This is possible because $r = |P - q| \neq 0$ for all points $P \in \Omega$ and thus the involved line integrals are always regular.

Subroutine STRESSB

This subroutine computes the stresses σ_x , σ_y and τ_{xy} at the boundary nodal points. First, it evaluates the derivatives $u_{,t}$ and $v_{,t}$ in the direction of the tangent to the boundary via finite differences (see Section 6.2.2), and then computes the stresses using Eqs. (7.118) and (7.120).

Subroutine STRESSIN

This subroutine computes the stresses σ_x , σ_y and τ_{xy} at the internal points on the basis of Eqs. (7.113).

Subroutine OUTPUTEL

This subroutine writes all the results in the output file.

The listing of program ELBECON is given below:

```
C====
       С
     PROGRAM ELBECON
С
С
C This program solves the two dimensional (EL)asticiy problem
C using the (B)oundary (E)lement method with (CON)stant
  boundary elements for domain with multiple boundaries
С
С
     USE MSIMSL
     IMPLICIT REAL*8 (A-H,O-Z)
     CHARACTER*15 INPUTFILE, OUTPUTFILE
С
C
  Set the maximum dimensions
С
     PARAMETER (N=44)
     PARAMETER (IN=3)
     PARAMETER (NB=1)
```

```
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```

```
С
С
   N= Number of boundary elements equal to number of boundary
С
       nodes
С
C IN= Number of internal points where the function u is calculated
С
C NB= Number of boundaries of the multiple boundary domain
С
       COMMON EL, GL, PN, ALAMDA
       DIMENSION NL(NB), KCODE(N)
       DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), G(2*N, 2*N), H(2*N, 2*N)
       DIMENSION UB(N), VB(N), TXB(N), TYB(N), A(2*N, 2*N), R(2*N)
       DIMENSION XIN(IN), YIN(IN), UIN(IN), VIN(IN)
       DIMENSION SXIN(IN), SYIN(IN), SXYIN(IN), SXB(N), SYB(N), SXYB(N)
С
С
   Read the names and open the input and output files
С
       WRITE (*,'(A)')' Name of the INPUTFILE (max.15 characters)'
             (*,'(A)') INPUTFILE
       READ
       WRITE (*, '(A)')' Name of the OUTPUTFILE (max.15 characters)'
             (*, '(A) ') OUTPUTFILE
       READ
       OPEN (1, FILE=INPUTFILE)
      OPEN (2, FILE=OUTPUTFILE)
С
   Read data from INPUTFILE
С
      CALL INPUTEL (XL, YL, XIN, YIN, KCODE, IPLANE, NL, UB, VB, N, IN, NB)
С
C
   Compute the G matrix
С
      CALL GMATREL(XL, YL, XM, YM, G, N, NL, NB)
С
С
   Compute the H matrix
С
      CALL HMATREL(XL, YL, XM, YM, H, N, NL, NB)
C
С
   Form the system of equations AX=B
C
      CALL ABMATREL (G, H, A, R, UB, VB, KCODE, N)
C
   Solve the system of equations
С
C
      CALL SOLVEQ(A, R, N, LSING)
С
C
   Form the vectors U and UN of all the boundary values
C
      CALL REORDEREL (R, UB, VB, TXB, TYB, KCODE, N)
С
C
   Compute the values of the displacements at the internal points
C
      CALL UVINTER (XL, YL, XIN, YIN, UB, VB, TXB, TYB, UIN, VIN, N, IN, NL, NB)
C
   Compute the values of the stresses at the internal points
С
С
      CALL STRESSIN(XL,YL,XIN,YIN,UB,VB,TXB,TYB,
     1
                           SXIN, SYIN, SXYIN, N, IN, NL, NB)
С
   Compute the values of the boundary stresses
C
С
      CALL STRESSB(XL, YL, UB, VB, TXB, TYB, SXB, SYB, SXYB, N, NL, NB)
С
С
   Print the results in the OUTPUTFILE
C
      CALL OUTPUTEL (XM, YM, UB, VB, TXB, TYB, XIN, YIN, UIN, VIN,
     1
                      SXB, SYB, SXYB, SXIN, SYIN, SXYIN, N, IN)
```

```
С
C Close input and output files
C
      CLOSE(1)
      CLOSE(2)
      STOP
      END
С
С
С
      SUBROUTINE INPUTEL(XL,YL,XIN,YIN,KCODE,IPLANE,NL,UB,VB,N,IN,NB)
С
C
С
  This subroutine reads the input data from the input file
С
   and writes them in the output file
С
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON EL, GL, PN, ALAMDA
      CHARACTER*80 NAME, TITLE
      DIMENSION NL (NB), KCODE (N)
      DIMENSION UB(N), VB(N)
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN)
С
      WRITE(2,100)
  100 FORMAT('',76('*'))
      READ(1, '(A) ') NAME
      WRITE(2,'(A)') NAME
С
С
  Read the title of the program
с
      READ(1, '(A)')TITLE
      WRITE(2, '(A) ') TITLE
С
      WRITE(2,200)N, IN, NB
  200 FORMAT(//'DATA'//2X,'NUMBER OF BOUNDARY ELEMENTS ='
            ,13/2X, 'NUMBER OF INTERNAL POINTS
                                              =',I3/2X,
     1
     1
             'NUMBER OF BOUNDARIES
                                         =',I3)
С
  Read the type of the problem: IPLANE=0 for plane strain,
С
С
                                IPLANE=1 for plane stress
С
      READ(1, *) IPLANE
      IF (IPLANE.EQ.0) THEN
      WRITE(2,700)
  700 FORMAT(/' PLANE STRAIN PROBLEM'/)
      ELSE
      WRITE(2,450)
  450 FORMAT(/' PLANE STRESS PROBLEM'/)
      ENDIF
С
C
  Read the elastic constants
С
      READ(1,*) EL, PN
С
      WRITE(2,150) EL, PN
  150 FORMAT(2X, 'ELASTIC CONSTANTS:', '
                                       ELASTIC MODULUS =', E10.4,
         ' POISSON RATIO =',F5.2/)
     1
С
      IF (IPLANE.EQ.0) THEN
      EL=EL/(1.-PN**2)
      PN=PN/(1-PN)
      GL*EL/(2.*(1+PN))
      ALAMDA=PN*EL/(1-PN**2)
      ELSEIF(IPLANE.NE.0) THEN
```

```
GL=EL/(2.*(1+PN))
      ALAMDA=PN*EL/(1-PN**2)
      ENDIF
С
   Read the of the last element at each boundary
С
с
      READ(1,*)(NL(I), I=1,NB)
С
  Read the coordinates of the extreme points of the boundary
С
С
   elements XL, YL
C
      READ(1,*) (XL(I),YL(I),I=1,N)
С
С
   Write the coordinates in the output file
C
      WRITE(2,300)
                 COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY',
  300 FORMAT(//'
               ' ELEMENTS',//1X,'NODE',9X,' XL',13X,' YL'/)
     1
     DO 20 I=1,N
   20 WRITE(2,400) I,XL(I),YL(I)
  400 FORMAT(2X, I3, 2(3X, E14.5))
С
  Read the boundary conditions and store in UB(I) and VB(I)
С
С
     KCODE(I)=1 when u and v are prescribed
С
С
     KCODE(I) = 2 when tx and ty are prescribed
C
     KCODE(1)=3 when u and ty are prescribed
С
    KCODE(I)=4 when tx and v are prescribed
С
      READ(1,*) (KCODE(I), UB(I), VB(I), I=1, N)
С
  Write the boundary conditions in the output file
C
С
      WRITE(2,500)
  500 FORMAT(//2X, 'BOUNDARY CONDITIONS'//2X, 'NODE', 6X, ' KCODE',
     1 20X, 'PRESCRIBED VALUES'/)
      DO 30 I=1,N
   30 WRITE(2,600) I,KCODE(I),UB(I), VB(I)
  600 FORMAT (2X, I3, 9X, I3, 8X, E14.5, 8X, E14.5)
С
  Read the coordinates of the internal points
С
С
      READ(1,*) (XIN(I),YIN(I),I=1,IN)
      RETURN
      END
С
С
      C==
C
      SUBROUTINE GMATREL (XL, YL, XM, YM, G, N, NL, NB)
С
С
С
   This subroutine computes the elements of the G matrix
С
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON EL, GL, PN, ALAMDA
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N)
      DIMENSION NL(NB),G(2*N,2*N)
С
   Compute the nodal coordinates and store them in the arrays XM
С
С
   and YM
С
      PI=ACOS(-1.)
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
```

```
DO 10 I=1,N
       XM(I) = (XL(I) + XL(I+1))/2,
   10 YM(I) = (YL(I) + YL(I+1))/2.
       IF(NB.LE.1) GOTO 40
       XM(NL(1)) = (XL(NL(1)) + XL(1))/2.
       YM(NL(1)) = (YL(NL(1)) + YL(1))/2.
       DO 15 K=2,NB
       XM(NL(K)) = (XL(NL(K)) + XL(NL(K-1)+1))/2.
   15 YM(NL(K)) = (YL(NL(K)) + YL(NL(K-1) + 1))/2.
С
С
   Compute the elements of matrix G
С
   40 DO 20 I=1,N
      X0 = XM(I)
      Y0=YM(I)
      DO 20 J=1,N
      Xl = XL(J)
      Yl=YL(J)
      X2 = XL(J+1)
      Y2 = YL(J+1)
      IF(NB.LE.1) GOTO 60
      IF(J.NE.NL(1))GOTO 50
            X2 = XL(1)
            Y_{2}=Y_{1}(1)
            GOTO 60
   50
           DO 30 K=2,NB
          IF(J.NE.NL(K))GOTO 30
            X2 = XL(NL(K-1)+1)
            Y2 = YL(NL(K-1)+1)
            GOTO 60
   30
            CONTINUE
   60
      IF(I.NE.J)THEN
      CALL RLINTG (X0, Y0, X1, Y1, X2, Y2, RES11, RES12, RES22)
      G(2*I-1, 2*J-1) = RES11
      G(2*I-1, 2*J) = RES12
      G(2*I, 2*J-1) = RES12
      G(2*1, 2*J) = RES22
      ELSEIF(I.EQ.J) THEN
      DX = X2 - X1
      DY = Y2 - Y1
      SL=DSQRT (DX**2+DY**2)
      COSA=DX/SL
      SINA=DY/SL
      SPIG=SL/(8.*PI*GL)
      RES11=-SPIG*((3.-PN)*(DLOG(SL/2.)-1.)~(1.+PN)*COSA**2)
      RES12=SPIG*(1.+PN)*COSA*SINA
      RES22=-SPIG*((3.-PN)*(DLOG(SL/2.)-1.)~(1.+PN)*SINA**2)
      G(2*I-1,2*J-1)=RES11
      G(2*I-1,2*J)=RES12
      G(2*I, 2*J-1) = RES12
      G(2*I, 2*J) = RES22
      ENDIF
   20 CONTINUE
      RETURN
      END
С
С
С
      SUBROUTINE RLINTG(X0,Y0,X1,Y1,X2,Y2,RES11,RES12,RES22)
С
С
с
  This subroutine computes the off-diagonal elements of the
С
  matrix G
C
```

```
Chapter 7
```

```
RA= The distance of the point 0 to the Gauss integration point
C
С
       on the boundary element
С
С
  WG= The weights of the Gauss integration
С
С
  XI= The coordinates of the Gauss integration points in the
с
       interval [-1,1]
С
  XC,YC=The global coordinates of the Gauss integration points
С
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON EL, GL, PN, ALAMDA
      DIMENSION XC(4),YC(4),XI(4),WG(4)
      DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      PI=ACOS(-1.)
      AX = (X2 - X1) / 2.
      AY = (Y2 - Y1) / 2.
      BX = (X2 + X1) / 2.
      BY = (Y2 + Y1) / 2.
С
С
  Compute the Jacobian
С
      SL=DSORT(AX**2+AY**2)
С
С
  Compute the line integral
С
      RES11=0.
      RES12=0.
      RES22=0.
С
      PIG=1./(8.*PI*GL)
      DO 30 I=1,4
      XC(I) = AX * XI(I) + BX
      YC(I) = AY * XI(I) + BY
      DX = XC(I) - X0
      DY = YC(I) - YO
      RA=DSQRT(DX**2+DY**2)
      COSA=DX/RA
      SINA=DY/RA
      RES11=RES11-PIG*((3.-PN)*DLOG(RA)-(1+PN)*COSA**2)*WG(I)*SL
      RES12=RES12+PIG*(1.+PN)*COSA*SINA*WG(I)*SL
   30 RES22=RES22-PIG*((3.-PN)*DLOG(RA)-(1+PN)*SINA**2)*WG(I)*SL
      RETURN
      END
С
С
C==
      С
      SUBROUTINE HMATREL(XL,YL,XM,YM,H,N,NL,NB)
С
С
  This subroutine computes the elements of the H matrix
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON EL, GL, PN, ALAMDA
      DIMENSION XL(N+1), YL(N+1), XM(N), YM(N), NL(NB)
      DIMENSION H(2*N, 2*N)
С
С
  Compute the nodal coordinates and store them in the arrays XM
С
  and YM
С
      PI=ACOS(-1.)
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
```

```
DO 10 I=1,N
      XM(I) = (XL(I) + XL(I+1))/2.
   10 YM(I) = (YL(I) + YL(I+1))/2.
      IF(NB.LE.1) GOTO 40
      XM(NL(1)) = (XL(NL(1)) + XL(1))/2.
      YM(NL(1)) = (YL(NL(1)) + YL(1))/2.
      DO 15 K=2,NB
      XM(NL(K)) = (XL(NL(K)) + XL(NL(K-1) + 1))/2.
   15 YM(NL(K)) = (YL(NL(K)) + YL(NL(K-1)+1))/2.
C
С
   Compute the elements of matrix H
С
   40 DO 20 I=1,N
      X0 = XM(I)
      Y0=YM(I)
      DO 20 J=1,N
      X1 = XL(J)
      Y1=YL(J)
      X2 = XL(J+1)
      Y2 = YL(J+1)
      IF(NB.LE.1) GOTO 60
      IF(J.NE.NL(1))GOTO 50
            X2 = XL(1)
            Y2 = YL(1)
            GOTO 60
           DO 30 K=2,NB
   50
          IF(J.NE.NL(K))GOTO 30
            X2 = XL(NL(K-1)+1)
            Y2 = YL(NL(K-1)+1)
            GOTO 60
   30
            CONTINUE
   60 IF(I.NE.J)THEN
      CALL RLINTH (X0, Y0, X1, Y1, X2, Y2, RES11, RES12, RES21, RES22)
      H(2*I-1,2*J-1)=RES11
      H(2*I-1, 2*J) = RES21
      H(2*I,2*J-1)=RES12
      H(2*I, 2*J) = RES22
      ELSEIF (I.EQ.J) THEN
      H(2*I-1, 2*J-1) = .5
      H(2*I-1,2*J)=0.
      H(2*I, 2*J-1) = 0.
      H(2*I, 2*J) = .5
      ENDIF
   20 CONTINUE
      RETURN
      END
с
С
C
      SUBROUTINE RLINTH(X0,Y0,X1,Y1,X2,Y2,RES11,RES12,RES21,RES22)
С
С
С
  This subroutine computes the off-diagonal elements of the
С
  matrix H
C
С
  RA= The distance of the point 0 to the Gauss integration point
C
       on the boundary element
С
С
  WG* The weights of the Gauss integration
С
С
  XI= The coordinates of the Gauss integration points in the
С
       interval [-1,1]
С
C XC, YC=The global coordinates of the Gauss integration points
```

```
С
       IMPLICIT REAL*8 (A-H,O-Z)
       COMMON EL, GL, PN, ALAMDA
       DIMENSION XC(4), YC(4), XI(4), WG(4)
       DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
       PI=ACOS(-1.)
      AX = (X2 - X1) / 2.
      AY = (Y2 - Y1) / 2.
      BX = (X2 + X1) / 2.
      BY = (Y2 + Y1) / 2.
      SL=DSQRT(AX**2+AY**2)
      ENX=AY/SL
      ENY=-AX/SL
С
С
   Compute the line integral
С
      RES11=0.
      RES12 \neq 0.
      RES21=0.
      RES22=0.
С
      A1=-(1.+PN)/(4.*PI)
      A2 = (1. - PN) / (1. + PN)
      DO 30 I=1,4
      XC(I) = AX + XI(I) + BX
      YC(I) = AY * XI(I) + BY
      DX = XC(1) - X0
      DY=YC(I)-Y0
      RA=DSQRT(DX**2+DY**2)
      COSA=DX/RA
      SINA=DY/RA
      RN=COSA*ENX+SINA*ENY
      RT=SINA*ENX-COSA*ENY
      RES11=RES11+(A1/RA)*(A2+2.*COSA**2)*RN*WG(I)*SL
      RES21=RES21+(A1/RA)*(2.*COSA*SINA*RN+A2*RT)*WG(I)*SL
      RES12=RES12+(A1/RA)*(2.*COSA*SINA*RN-A2*RT)*WG(I)*SL
   30 RES22=RES22+(A1/RA)*(A2+2.*SINA**2)*RN*WG(I)*SL
      RETURN
      END
С
C
С
      SUBROUTINE ABMATREL (G, H, A, R, UB, VB, KCODE, N)
С
С
С
  This subroutine rearranges the matrices G and H and produces the
С
  matrices A and R
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION G(2*N, 2*N), H(2*N, 2*N), A(2*N, 2*N)
      DIMENSION UB(N), VB(N), KCODE(N), R(2*N)
С
C
  Reorder the columns the system of equations and store them in A
С
      N2=2*N
      DO 20 I=1,N2
      R(I) = 0.
      DO 20 J=1,N2
   20 A(I,J)=0.
      DO 70 J=1,N
      GOTO (30, 40, 50, 60) KCODE (J)
   30 DO 35 I=1,N2
      A(I, 2*J-1) = -G(I, 2*J-1)
```

```
A(I, 2*J) = -G(I, 2*J)
   35 R(I) = R(I) - H(I, 2*J-1) * UB(J) - H(I, 2*J) * VB(J)
      GOTO 70
   40 DO 45 I=1,N2
      A(I,2*J-1)=H(I,2*J-1)
      A(I,2*J) = H(I,2*J)
   45 R(I) = R(I) + G(I, 2 + J - 1) + UB(J) + G(I, 2 + J) + VB(J)
      GOTO 70
   50 DO 55 I=1,N2
      A(I, 2*J-1) = -G(I, 2*J-1)
      A(I,2*J) = H(I,2*J)
   55 R(I)=R(I)-H(I,2*J-1)*UB(J)+G(I,2*J)*VB(J)
      GOTO 70
   60 DO 65 I=1,N2
      A(I,2*J-1)=H(I,2*J-1)
      A(I,2*J) = -G(I,2*J)
   65 R(I) = R(I) + G(I, 2*J-1) * UB(J) - H(I, 2*J) * VB(J)
   70 CONTINUE
      RETURN
      END
С
С
        Car
С
      SUBROUTINE SOLVEQ(A, R, N, LSING)
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(2*N, 2*N), R(2*N)
     NN = 2 * N
      CALL LEQS(A, R, NN, LSING)
      IF(LSING.EQ.0)THEN
     WRITE(2,150)
  150 FORMAT(/,'',76('*')//2X'The system has been solved regularly'/)
      ELSEIF(LSING.EQ.1) THEN
      WRITE(2,170)
  170 FORMAT(/,'',76('*')//2X'The system is singular'/)
      ENDIF
      RETURN
      END
С
С
C====
           C
      SUBROUTINE LEQS(A, B, N, LSING)
С
С
    This subroutine uses Gauss elimination to solve
С
    a system of linear equations, [A] \{X\} = \{B\}, where
С
    A : One-dimensional array which contains the occasional row of
С
С
        the two-dimensional array of the coefficients of the unknowns
С
    B : One-dimensional array which contains the known coefficients
C
    N : Integer denoting the number of the unknowns
С
    LSING: Integer taking the values:
          LSING = 0, if the system has been solved regularly
С
          LSING = 1, if the system is singular
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(1), B(1)
С
      LSING=0
      JJ = -N
      DO 10 J=1,N
        JY=J+1
        JJ=JJ+N+1
```

```
AMAX=0.0
        IHELP=JJ-J
      DO 20 I=J,N
        IJ=IHELP+I
        IF (ABS (AMAX) - ABS (A (IJ))) 30,20,20
   30
           AMAX=A(IJ)
           IMAX≖I
   20 CONTINUE
         IF (ABS (AMAX).EQ.0.) THEN
            LSING=1
            RETURN
         END IF
      I1=J+N*(J-2)
      IHELP=IMAX-J
      DO 40 K=J,N
        I1=I1+N
        I2=I1+IHELP
        ATEMP=A(I1)
        A(I1) = A(I2)
        A(I2) = ATEMP
   40 A(I1) = A(I1) / AMAX
      ATEMP=B(IMAX)
      B(IMAX) = B(J)
      B(J) = ATEMP/AMAX
         IF(J-N) 50,70,50
   50
            IQS=N*(J-1)
            DO 10 IX=JY,N
              IXJ=IQS+IX
              IHELP=J-IX
              DO 60 JX=JY,N
                IJREF=N*(JX-1)+IX
                JJX=IJREF+IHELP
   60
                A(IJREF) = A(IJREF) - (A(IXJ) * A(JJX))
   10 B(IX) = B(IX) - B(J) * A(IXJ)
   70
         NY = N - 1
      NN = N * N
      DO 80 J=1,NY
        I1=NN-J
        T2=N-J
        I3=N
      DO 80 K=1,J
        B(I2) = B(I2) - A(I1) * B(I3)
        I1=I1-N
   80
        13=13-1
      RETURN
      END
С
с
C===
         С
      SUBROUTINE REORDEREL (R, UB, VB, TXB, TYB, KCODE, N)
С
С
  This subroutine rearranges the arrays UB and VB
С
С
  in such a way that all boundary values of u and v
  are stored in UB and VB, while all boundary
С
С
  values of tx and ty in TXB and TYB
С
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION UB(N), VB(N), R(2*N), TXB(N), TYB(N), KCODE(N)
С
      DO 50 I=1,N
      GOTO(10,20,30,40) KCODE(I)
   10 TXB(I) = R(2*I-1)
      TYB(I) = R(2 * I)
```

```
GOTO 50
   20 TXB(I) = UB(I)
      TYB(I) = VB(I)
      UB(I) = R(2 + I - 1)
      VB(I) = R(2*I)
      GOTO 50
   30 TYB(I)=VB(I)
      TXB(I) = R(2 + I - 1)
      VB(I) = R(2 * I)
      GOTO 50
   40 TXB(I)=UB(I)
       UB(I) = R(2 + I - 1)
      TYB(I) = R(2 * I)
   50 CONTINUE
С
      RETURN
      END
С
С
C=
              С
      SUBROUTINE UVINTER(XL,YL,XIN,YIN,UB,VB,TXB,TYB,UIN,VIN,N,IN,NL,NB)
С
С
С
   This subroutine computes the values of the displacements u and v
С
   at the internal points
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), UB(N), VB(N)
      DIMENSION TXB(N), TYB(N), UIN(IN), VIN(IN)
      DIMENSION NL(NB)
С
С
   Compute the values of u and v at the internal points
C
      IF(NB.GT.1) GOTO 5
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
С
    5 DO 10 KK=1, IN
      UIN(KK) = 0.
      VIN(KK) = 0.
      DO 20 J=1,N
      X0=XIN(KK)
      Y0=YIN(KK)
      X1=XL(J)
      Y1=YL(J)
      X2 = XL(J+1)
      Y2 = YL(J+1)
      IF(NB.LE.1)GOTO 60
        IF(J.NE.NL(1))GOTO 50
            X2 = XL(1)
            Y2=YL(1)
            GOTO 60
   50
           DO 30 K=2,NB
          IF(J.NE.NL(K))GOTO 30
            X2 = XL(NL(K-1)+1)
            Y2 = YL(NL(K-1)+1)
            GOTO 60
   30
            CONTINUE
   60 CALL RLINTG (X0, Y0, X1, Y1, X2, Y2, G11, G12, G22)
      CALL RLINTH (X0, Y0, X1, Y1, X2, Y2, H11, H12, H21, H22)
      UIN (KK) =UIN (KK) -H11*UB (J) -H21*VB (J) +G11*TXB (J) +G12*TYB (J)
   20 VIN(KK) = VIN(KK) - H12*UB(J) - H22*VB(J) + G12*TXB(J) + G22*TYB(J)
   10 CONTINUE
С
```

```
RETURN
      END
С
С
C=
             C
      SUBROUTINE STRESSIN(XL,YL,XIN,YIN,UB,VB,TXB,TYB,
     1
                          SXIN, SYIN, SXYIN, N, IN, NL, NB)
С
С
С
   This subroutine computes the stresses at the internal points
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN), UB(N), VB(N)
      DIMENSION TXB(N), TYB(N)
      DIMENSION SXIN(IN), SYIN(IN), SXYIN(IN)
      DIMENSION NL(NB)
С
      IF(NB.GT.1) GOTO 5
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
С
    5 DO 10 KK=1, IN
      SXIN(KK) = 0.
      SYIN(KK) = 0.
      SXYIN(KK) = 0
     DO 20 J=1.N
     X0=XIN(KK)
     Y0=YIN(KK)
     X1 = XL(J)
     Y1 = YL(J)
     X2 = XL(J+1)
     Y2=YL(J+1)
      IF (NB.LE.1) GOTO 60
        IF(J.NE.NL(1))GOTO 50
           X2 = XL(1)
           Y2 = YL(1)
           GOTO 60
  50
          DO 30 K=2,NB
          IF(J.NE.NL(K))GOTO 30
            X2 = XL(NL(K-1)+1)
            Y2 = YL(NL(K-1)+1)
            GOTO 60
   30
            CONTINUE
   60 CALL SLINTH(X0,Y0,X1,Y1,X2,Y2,RES11,RES21,RES31,RES12,RES22,
    1
                 RES32, RESB11, RESB21, RESB31, RESB12, RESB22, RESB32)
С
      SXIN(KK) = SXIN(KK) + RES11*TXB(J) + RES12*TYB(J)
     1
                     -RESB11*UB(J)-RESB12*VB(J)
      SYIN(KK) = SYIN(KK) + RES21*TXB(J) + RES22*TYB(J)
     1
                     -RESB21*UB(J)-RESB22*VB(J)
   20 SXYIN(KK) = SXYIN(KK) + RES31*TXB(J) + RES32*TYB(J)
                       -RESB31*UB(J)-RESB32*VB(J)
     1
   10 CONTINUE
      RETURN
      END
С
С
C=
              С
      SUBROUTINE SLINTH (X0,Y0,X1,Y1,X2,Y2,RES11,RES21,RES31,RES12,
     1
              RES22, RES32, RESB11, RESB21, RESB31, RESB12, RESB22, RESB32)
С
C
С
   This subroutine computes the line integrals of the kernels in the
```

```
integral representations of the stresses
С
С
C RA= The distance of the point 0 to the Gauss integration point
С
       on the boundary element
С
С
  WG= The weights of the Gauss integration
С
С
  XI= The coordinates of the Gauss integration points in the
С
       interval [-1,1]
С
С
  XC,YC=The global coordinates of the Gauss integration points
С
С
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON EL, GL, PN, ALAMDA
      DIMENSION XC(4),YC(4),XI(4),WG(4)
      DATA XI/-0.86113631,-0.33998104,0.33998104,0.86113631/
      DATA WG/0.34785485,0.65214515,0.65214515,0.34785485/
      PI=ACOS(-1.)
      AX = (X2 - X1) / 2.
      AY = (Y2 - Y1) / 2.
      BX = (X2 + X1) / 2.
      BY = (Y2 + Y1) / 2.
      SL=DSORT(AX**2+AY**2)
      ENX=AY/SL
      ENY=-AX/SL
с
С
  Compute the line integral
С
      RES11=0.
      RES21=0.
      RES31=0.
      RES12=0.
      RES22=0.
      RES32=0.
      RESB11=0.
      RESB21=0.
      RESB31=0.
      RESB12=0.
      RESB22=0.
      RESB32=0.
С
      Al=-(1.+PN)/(4.*PI)
      A2 = (1. - PN) / (1. + PN)
      A3=-2.*GL*A1
      DO 30 T=1.4
      XC(I) = AX * XI(I) + BX
      YC(I) = AY * XI(I) + BY
      DX = XC(I) - X0
      DY=YC(I)-Y0
      RA=DSQRT (DX**2+DY**2)
      COSA=-DX/RA
      SINA=-DY/RA
      RN=COSA*ENX+SINA*ENY
С
      RES11=RES11+(A1/RA)*(A2*COSA+2.*COSA**3)*WG(I)*SL
      RES21=RES21+(A1/RA)*(-A2*COSA+2.*COSA*SINA**2)*WG(I)*SL
      RES31=RES31+(A1/RA)*(A2*SINA+2.*COSA**2*SINA)*WG(I)*SL
      RES12=RES12+(A1/RA)*(-A2*SINA+2.*COSA**2*SINA)*WG(I)*SL
      RES22=RES22+(A1/RA)*(A2*SINA+2.*SINA**3)*WG(I)*SL
      RES32=RES32+(A1/RA)*(A2*COSA+2.*COSA*SINA**2)*WG(I)*SL
С
      RESB11=RESB11+(A3/RA**2)*(2.*COSA*RN*(1.-4.*COSA**2)+
     1
                                 (2.*COSA**2+1.)*ENX)*WG(I)*SL
```

```
RESB21=RESB21+(A3/RA**2)*(-8.*SINA**2*COSA*RN+2.*COSA*SINA*ENY+
     1
                                                 ENX) *WG(I) *SL
      RESB12=RESB12+(A3/RA**2)*(-8.*COSA**2*SINA*RN+2.*COSA*SINA*ENX+
     1
                                                 ENY) *WG(I) *SL
      RESB22=RESB22+(A3/RA**2)*(2.*SINA*RN*(1.-4.*SINA**2)+
     1
                                (2.*SINA**2+1.)*ENY)*WG(I)*SL
   30 CONTINUE
      RESB31=RESB12
      RESB32=RESB21
      RETURN
      END
С
С
C=:
         С
      SUBROUTINE STRESSB(XL,YL,UB,VB,TXB,TYB,SXB,SYB,SXYB,N,NL,NB)
С
С
C
   This subroutine computes the stress at the boundary
С
   nodal points
С
      IMPLICIT REAL*8 (A-H, O-Z)
      COMMON EL, GL, PN, ALAMDA
      DIMENSION XL(N+1), YL(N+1), UB(N), VB(N), TXB(N), TYB(N)
      DIMENSION SXB(N), SYB(N), SXYB(N), SL(N), ENX(N), ENY(N), A(4,4), B(4)
      DIMENSION NL(NB)
С
      IF(NB.GT.1) GOTO 5
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
С
С
  Computation of the element half lengths and direction
С
  cosines of the normal vector
С
    5 DO 10 K=1,NB
      IF(K.EQ.1) THEN
      NF \simeq 1
      ELSE
      NF = NL(K-1) + 1
      ENDIF
      DO 10 I=NF, NL(K)
      IF(I.EQ.NL(K))THEN
      AX = (XL(NF) - XL(NL(K)))/2.
      AY = (YL(NF) - YL(NL(K)))/2.
      SL(I)=DSQRT(AX**2+AY**2)
      ENX(I) = AY/SL(I)
      ENY(I) = -AX/SL(I)
      ELSE
      AX = (XL(I+1) - XL(I))/2.
      AY = (YL(I+1) - YL(I))/2.
      SL(I) = DSQRT(AX * 2 + AY * 2)
      ENX(I) = AY/SL(I)
      ENY(I) = -AX/SL(I)
      ENDIF
   10 CONTINUE
С
С
   Computation of the tangential derivatives UT and VT
С
      DO 20 K=1,NB
      IF(K.EQ.1)THEN
      NF=1
      ELSE
      NF = NL(K - 1) + 1
```

```
ENDIF
       DO 20 I=NF, NL(K)
       IF (I.EQ.NF) THEN
       S1=SL(NL(K))+SL(NF)
       S2=SL(NF)+SL(NF+1)
       U1=UB(NL(K))
       U2=UB(NF)
       U3=UB(NF+1)
       V1=VB(NL(K))
       V2 = VB(NF)
       V3 = VB(NF+1)
       ELSE IF (I.EQ.NL(K)) THEN
       Sl=SL(NL(K)-1)+SL(NL(K))
       S2 = SL(NL(K)) + SL(NF)
       U1=UB(NL(K)-1)
      U2 = UB(NL(K))
       U3=UB(NF)
       V1 = VB(NL(K) - 1)
      V2 = VB(NL(K))
      V3=VB(NF)
      ELSE
       S1=SL(I-1)+SL(I)
      S2=SL(I)+SL(I+1)
      U1 = UB(I-1)
      U2 = UB(I)
      U3=UB(I+1)
      V1=VB(I-1)
      V2 = VB(I)
      V3=VB(I+1)
      ENDIF
с
С
   Computation of the boundary stresses
с
      UT= (S1**2*U3-S2**2*U1+(S2**2-S1**2)*U2)
     1
             /(S1*S2*(S1+S2))
      VT=(S1**2*V3-S2**2*V1+(S2**2-S1**2)*V2)
             /(S1*S2*(S1+S2))
     1
      A(1,1) = (ALAMDA+2.*GL) *ENX(I)
      A(1,2) = GL + ENY(1)
      A(1,3)=GL*ENY(1)
      A(1,4) = ALAMDA*ENX(I)
      A(2,1) = ALAMDA*ENY(1)
      A(2,2) = GL + ENX(I)
      A(2,3) = GL \times ENX(I)
      A(2,4) = (ALAMDA+2.*GL) * ENY(1)
      A(3,1) = -ENY(1)
      A(3,2) = ENX(I)
      A(3,3) = 0.
      A(3,4) = 0.
      A(4,1) = 0.
      A(4,2) = 0.
      A(4,3) = -ENY(1)
      A(4,4) = ENX(1)
      B(1) = TXB(I)
      B(2) = TYB(I)
      B(3) = UT
      B(4) = VT
      CALL LEQS(A, B, 4, LSING)
      SXB(I) = ALAMDA*(B(1)+B(4))+2.*GL*B(1)
      SYB(1) = ALAMDA * (B(1) + B(4)) + 2 * GL * B(4)
      SXYB(I) = GL * (B(2) + B(3))
   20 CONTINUE
      RETURN
       END
```

```
С
C = =
          С
     SUBROUTINE OUTPUTEL (XM, YM, UB, VB, TXB, TYB, XIN, YIN, UIN, VIN,
     1
                         SXB, SYB, SXYB, SXIN, SYIN, SXYIN, N, IN)
С
С
С
  This subroutine prints the results in the outputfile.
С
      IMPLICIT REAL*8 (A-H,O-Z)
     DIMENSION XM(N), YM(N), UB(N), VB(N), TXB(N), TYB(N)
     DIMENSION XIN(N), YIN(N), UIN(IN), VIN(IN)
     DIMENSION SXB(N), SYB(N), SXYB(N)
     DIMENSION SXIN(IN), SYIN(IN), SXYIN(IN)
C
     WRITE(2,100)
  100 FORMAT('',76('*')///'RESULTS'///2X,'BOUNDARY NODES'//'NODE',
              5x, 'X', 11X, 'Y', 12X, 'U', 11X, 'V', 10X, 'TXB', 9X, 'TYB'/)
    1
     DO 10 I=1,N
  10 WRITE(2,200)I,XM(I),YM(I),UB(I),VB(I),TXB(I),TYB(I)
  200 FORMAT(I3,2(1X,E11.5),1X,4(1X,E11.5))
С
     WRITE (2,300)
 300 FORMAT(//,2X,'DISPLACEMENTS AT INTERNAL POINTS'//' POINT',9X,
              'X',14X,'Y',15X,'U',15X,'V'/)
    1
     DO 20 K=1,IN
  20 WRITE(2,400)K,XIN(K),YIN(K),UIN(K),VIN(K)
 400 FORMAT(15,4(2X,E14.5))
     WRITE(2,600)
 600 FORMAT(//,2X,'STRESSES AT THE BOUNDARY NODAL POINTS'/)
     WRITE(2,700)
 700 FORMAT('NODE',7X,'X',14X,'Y',11X,'SXB',12X,'SYB',10X,'SXYB'/)
     DO 30 I=1,N
  30 WRITE(2,800)I,XM(I),YM(I),SXB(I),SYB(I),SXYB(I)
 800 FORMAT(13,5(2X,E12.5))
     WRITE (2,900)
 900 FORMAT(//,2X,'STRESSES AT THE INTERNAL POINTS'/)
     WRITE (2,1000)
1000 FORMAT('NODE', 7X, 'X', 14X, 'Y', 11X, 'SXIN', 10X, 'SYIN', 10X, 'SXYIN'/)
     DO 40 I=1, IN
  40 WRITE(2,800)I,XIN(I),YIN(I),SXIN(I),SYIN(I),SXYIN(I)
     WRITE(2,500)
 500 FORMAT(/,'',76('*'))
     RETURN
     END
С
C
C≠
```

Example 7.2

The scope of this example is to illustrate the use of program ELBECON by solving a simple plane stress problem. The body under consideration is a deep beam (i.e. the length of the beam is not large in comparison with its depth) clamped at its two ends. It has a thickness is h = 0.1 m and material constants $E = 2 \times 10^5$ kN/m², $\nu = 0.20$. All the other data are shown in Fig. 7.9.



Figure 7.9 Deep beam clamped at both ends.

The results are obtained using a total of N=44 boundary elements. The horizontal boundaries (upper and lower) are divided each into NX=15 boundary elements, whereas the vertical ones (left and right) into NY=7 elements each. The data file for elasticity problems of rectangular domains can be constructed automatically by program RECT-4.FOR which has been written for this purpose. This program is listed below for the values N=44, NX=15, NY=7, IN=3, NB=1, IPLANE=1, IX=1, JY=3 of Example 7. 2.

```
C====
                     С
      PROGRAM RECT4
С
  This program creates the INPUTFILE for ELBECON when the
С
C domain is a rectangle AA*BB
С
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*15 INPUTFILE
      CHARACTER*80 NAME, TITLE
      PARAMETER (N=44)
      PARAMETER (IN=3)
      PARAMETER (NB=1)
      DIMENSION XL(N+1), YL(N+1), XIN(IN), YIN(IN)
     DIMENSION KCODE(N), UB(N), VB(N), NL(NB)
с
      WRITE (*,'(A)')' Name of the INPUTFILE (max.15 characters)'
     READ (*,'(A)') INPUTFILE
      OPEN (1, FILE=INPUTFILE)
      WRITE (*,'(A)')' User NAME (max.15 characters)'
      READ(*, '(A)')NAME
      WRITE(1, '(A) ')NAME
      WRITE (*,'(A)')' Program TITLE (max.15 characters)'
      READ(*, '(A)')TITLE
      WRITE(1, '(A) ') TITLE
```

```
WRITE(1,100)
  100 FORMAT(' ',79(' '))
C
       IPLANE=1
       WRITE(1,150)IPLANE
  150 FORMAT(14)
      EL=200000.
      PN=.2
      WRITE(1,200)EL,PN
  200 FORMAT(F10.0, F6.3)
      WRITE(1,100)
С
      NL(1) = N
      WRITE(1,250)(NL(I),I=1,NB)
  250 FORMAT(14)
С
С
   Boundary coordinates
с
      AA=3.
      BB=1.
      NX = 15
      NY = 7
      DA=AA/NX
      DB=BB/NY
      DO 1 I=1,NX
      XL(I) = (I-1) * DA
      YL(I) = 0.
    1 WRITE(1,300)XL(I),YL(I)
  300 FORMAT(2(F14.8,2X))
С
      DO 2 I=1,NY
      XL(I+NX) = AA
      YL(I+NX) = (I-1) *DB
    2 WRITE(1,300)XL(I+NX),YL(I+NX)
с
      DO 3 I=1,NX
      XL(I+NX+NY) = AA - (I-1) * DA
      YL(I+NX+NY) = BB
    3 WRITE(1,300)XL(I+NX+NY),YL(I+NX+NY)
С
      DO 4 I=1,NY
      XL(I+NX+NY+NX) = 0.
      YL(I+NX+NY+NX) = BB-(I-1)*DB
    4 WRITE(1,300)XL(I+NX+NY+NX),YL(I+NX+NY+NX)
      WRITE(1,100)
С
С
  Boundary conditions
С
      XL(N+1) = XL(1)
      YL(N+1) = YL(1)
      DO 10 I=1,NX
      KCODE(I) = 2
      XM = (XL(I) + XL(I+1))/2.
      YM = (YL(I) + YL(I+1))/2.
      UB(I)=0.
   10 VB(I) = 0.
      DO 20 I=1,NY
      II = I + NX
      KCODE(II)=1
      XM = (XL(II) + XL(II+1))/2.
      YM = (YL(II) + YL(II+1))/2.
      UB(II)=0.
   20 VB(II) = 0.
      DO 30 I=1,NX
       II = I + NX + NY
```

```
KCODE(II) = 2
      XM = (XL(II) + XL(II+1))/2.
      YM = (YL(II) + YL(II+1))/2.
      UB(II) = 0.
   30 VB(II) = -1.
      DO 40 I=1,NY
      II = I + NX + NY + NX
      KCODE(II) = 1
      XM = (XL(II) + XL(II+1))/2.
      YM = (YL(II) + YL(II+1))/2.
      UB(II) = 0.
   40 VB(II)=0.
      DO 50 I=1,N
   50 WRITE(1,400)KCODE(I),UB(I),VB(I)
  400 FORMAT(14,3X,2(F18.8))
С
      WRITE(1,100)
С
С
   Coordinates of the internal points
С
      IX=1
      JY=3
      DX=AA/IX
      DY=BB/(JY+1)
      WRITE(1,100)
С
     DO 5 J=1,JY
      DO 5 I=1,IX
      K = (J-1) * JY + I
      XIN(K) = DX/2. + (I-1) * DX
      YIN(K) = DY + (J-1) * DY
      WRITE(1,300)XIN(K),YIN(K)
    5 CONTINUE
С
      WRITE(1,100)
С
      STOP
      END
С
```

EXAMPLE 7.2 (DATA)

J.T.KATSIKADELIS EXAMPLE 7.2

> 1 200000. .200

44

.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000
.00000000

2.40000000	.00000000	
2,60000000	. 00000000	
2 80000000	00000000	
2.00000000		
3.00000000	.00000000	
3.00000000	.14285714	
3.00000000	.28571429	
3.00000000	.42857143	
3.00000000	.57142857	
3 00000000	71428571	
3.00000000	./11203/1	
3.00000000	.85/14286	
3.00000000	1.00000000	
2.80000000	1.00000000	
2.60000000	1.00000000	
2.40000000	1.00000000	
2,20000000	1,0000000	
2 00000000	1 00000000	
1.00000000	1.00000000	
1.80000000	1.0000000	
1.60000000	1.00000000	
1.40000000	1.00000000	
1.20000000	1.00000000	
1.00000000	1.00000000	
80000000	1 0000000	
60000000	1 00000000	
.00000000	1.00000000	
.40000000	1.00000000	
.20000000	1.00000000	
.00000000	1.00000000	
.00000000	.85714286	
.00000000	.71428571	
.00000000	57142857	
00000000	42957142	
	.1205/115	
.00000000	.285/1429	
.00000000	.14285/14	
2	.0000000	.00000000
2 2	.00000000	.00000000
2 2 2	.0000000 .0000000 .0000000	.00000000 .0000000 .0000000
2 2 2 2	.0000000 .0000000 .0000000 .0000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2	.00000000 .00000000 .0000000 .0000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2	.00000000 .00000000 .00000000 .00000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2 2	.00000000 .00000000 .00000000 .00000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2	.00000000 .0000000 .0000000 .0000000 .000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .00000000 .00000000 .00000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .00000000 .00000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.00000000 .0000000 .0000000 .0000000 .000000	
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2		.00000000 .0000000 .0000000 .0000000 .000000
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2		
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	.0000000 .0000000 .0000000 .0000000 .000000	.00000000 .0000000 .0000000 .0000000 .000000

2		.00000000	-1.00000000
2		.00000000	-1.00000000
2		.00000000	-1.00000000
2		.0000000	-1.00000000
2		.0000000	-1.00000000
1		.0000000	.00000000
1		.00000000	.00000000
1		.0000000	.00000000
1		.0000000	.00000000
1		.0000000	.00000000
1		.00000000	.00000000
1		.0000000	.00000000
1	50000000	2500000	

1.50000000	.25000000
1.50000000	.50000000
1.50000000	.75000000

EXAMPLE 7.2 (RESULTS)

DATA

NUMBER	OF	BOUNDARY	ELEMENTS	=	44
NUMBER	OF	INTERNAL	POINTS	=	3
NUMBER	OF	BOUNDARIE	s	=	1

PLANE STRESS PROBLEM

ELASTIC CONSTANTS: ELASTIC MODULUS = .2000E+06 POISSON RATIO = .20

COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELEMENTS

NODE	XL	YL
1	.00000E+00	.00000E+00
2	.20000E+00	.00000E+00
3	.40000E+00	.00000E+00
4	.60000E+00	.00000E+00
5	.80000E+00	.00000E+00
6	.10000E+01	.00000E+00
7	.12000E+01	.00000E+00
8	.14000E+01	.00000E+00
9	.16000E+01	.00000E+00
10	.18000E+01	.00000E+00
11	.20000E+01	.00000E+00
12	.22000E+01	.00000E+00
13	.24000E+01	.00000E+00
14	.26000E+01	.00000E+00
15	.28000E+01	.00000E+00
16	.30000E+01	.00000E+00
17	.30000E+01	.14286E+00
18	.30000E+01	.28571E+00
19	.30000E+01	.42857E+00
20	.30000E+01	.57143E+00
21	.30000E+01	.71429E+00

22	.30000E+01	.85714E+00
23	.30000E+01	.10000E+01
24	.28000E+01	.10000E+01
25	.26000E+01	.10000E+01
26	.24000E+01	.10000E+01
27	.22000E+01	.10000E+01
28	.20000E+01	.10000E+01
29	.18000E+01	.10000E+01
30	.16000E+01	.10000E+01
31	.14000E+01	.10000E+01
32	.12000E+01	.10000E+01
33	.10000E+01	.10000E+01
34	.80000E+00	.10000E+01
35	.60000E+00	.10000E+01
36	.40000E+00	.10000E+01
37	.20000E+00	.10000E+01
38	.00000E+00	.10000E+01
39	.00000E+00	.85714E+00
40	.00000E+00	.71429E+00
41	.00000E+00	.57143E+00
42	.00000E+00	.42857E+00
43	.00000E+00	.28571E+00
44	.00000E+00	.14286E+00

BOUNDARY CONDITIONS

NODE	KCODE	PRESCRI	BED VALUES
1	2	.00000E+00	.00000E+00
2	2	.00000E+00	.00000E+00
3	2	.00000E+00	.00000E+00
4	2	.00000E+00	.00000E+00
5	2	.00000E+00	.00000E+00
6	2	.00000E+00	.00000E+00
7	2	.00000E+00	.00000E+00
8	2	.00000E+00	.00000E+00
9	2	.00000E+00	.00000E+00
10	2	.00000E+00	.00000E+00
11	2	.00000E+00	.00000E+00
12	2	.00000E+00	.00000E+00
13	2	.00000E+00	.00000E+00
14	2	.00000E+00	.00000E+00
15	2	.00000E+00	.00000E+00
16	1	.00000E+00	.00000E+00
17	1	.00000E+00	.00000E+00
18	1	.00000E+00	.00000E+00
19	1	.00000E+00	.00000E+00
20	1	.00000E+00	.00000E+00
21	1	.00000E+00	.00000E+00
22	1	.00000E+00	.00000E+00
23	2	.00000E+00	10000E+01
24	2	.00000E+00	10000E+01
25	2	.00000E+00	10000E+01
26	2	.00000E+00	10000E+01
27	2	.00000E+00	10000E+01
28	2	.00000E+00	10000E+01
29	2	.00000E+00	10000E+01
30	2	.00000E+00	10000E+01
31	2	.00000E+00	10000E+01
32	2	.00000E+00	10000E+01
33	2	.00000E+00	10000E+01
34	2	.00000E+00	10000E+01
35	2	.00000E+00	10000E+01
36	2	.00000E+00	10000E+01

270 BOUNDARY ELEMENTS

37	2	.00000E+00	10000E+01	
38	1	.00000E+00	.00000E+00	
39	1	.00000E+00	.00000E+00	
40	1	.00000E+00	.00000E+00	
41	1	.00000E+00	.00000E+00	
42	1	.00000E+00	.00000E+00	
43	1	.00000E+00	.00000E+00	
44	1	.00000E+00	.00000E+00	
42 43 44	1 1 1	.00000E+00 .00000E+00 .00000E+00	.00000E+00 .00000E+00 .00000E+00	

The system has been solved regularly

RESULTS

BOUNDARY NODES

NODE	x	Y	υ	v	TXB	TYB
1	.10000E+00	.00000E+00	11073E-05	47147E-06	.00000E+00	.00000E+00
2	.30000E+00	.00000E+00	24208E-05	22679E-05	.00000E+00	.00000E+00
3	.50000E+00	.00000E+00	30557E-05	44605E-05	.00000E+00	.00000E+00
4	.70000E+00	.00000E+00	31222E-05	66588E-05	.00000E+00	.00000E+00
5	.90000E+00	.00000E+00	27353E-05	86054E-05	.00000E+00	.00000E+00
6	.11000E+01	.00000E+00	20100E-05	10118E-04	.00000E+00	.00000E+00
7	.13000E+01	.00000E+00	10607E-05	11073E-04	.00000E+00	.00000E+00
8	.15000E+01	.00000E+00	74751E-21	11399E-04	.00000E+00	.00000E+00
9	.17000E+01	.00000E+00	.10607E-05	11073E-04	.00000E+00	.00000E+00
10	.19000E+01	.00000E+00	.20100E~05	10118E-04	.00000E+00	.00000E+00
11	.21000E+01	.00000E+00	.27353E-05	86054E-05	.00000E+00	.00000E+00
12	.23000E+01	.00000E+00	.31222E-05	66588E-05	.00000E+00	.00000E+00
13	.25000E+01	.00000E+00	.30557E-05	44605E-05	.00000E+00	.00000E+00
14	.27000E+01	.00000E+00	.24208E-05	22679E-05	.00000E+00	.00000E+00
15	.29000E+01	.00000E+00	.11073E-05	47147E-06	.00000E+00	.00000E+00
16	.30000E+01	.71429E-01	.00000E+00	.00000E+00	20636E+01	54870E+00
17	.30000E+01	.21429E+00	.00000E+00	.00000E+00	91311E+00	.30483E-01
18	.30000E+01	.35714E+00	.00000E+00	.00000E+00	51615E+00	.19079E+00
19	.30000E+01	.50000E+00	.00000E+00	.00000E+00	19883E+00	.29576E+00
20	.30000E+01	.64286E+00	.00000E+00	.00000E+00	.13260E+00	.34608E+00
21	.30000E+01	.78571E+00	.00000E+00	.00000E+00	.57485E+00	.34953E+00
22	.30000E+01	.92857E+00	.00000E+00	.00000E+00	.22960E+01	.39736E+00
23	.29000E+01	.10000E+01	15265E-05	16042E-05	.00000E+00	10000E+01
24	.27000E+01	.10000E+01	27364E-05	42788E-05	.00000E+00	10000E+01
25	.25000E+01	.10000E+01	31999E-05	67858E-05	.00000E+00	10000E+01
26	.23000E+01	.10000E+01	31621E-05	90767E-05	.00000E+00	10000E+01
27	.21000E+01	.10000E+01	27298E-05	11037E-04	.00000E+00	10000E+01
28	.19000E+01	.10000E+01	19935E-05	12543E-04	.00000E+00	10000E+01
29	.17000E+01	.10000E+01	10495E-05	13491E-04	.00000E+00	10000E+01
30	.15000E+01	.10000E+01	.47646E-21	13814E-04	.00000E+00	10000E+01
31	.13000E+01	.10000E+01	.10495E-05	13491E-04	.00000E+00	10000E+01
32	.11000E+01	.10000E+01	.19935E-05	12543E-04	.00000 E +00	10000E+01
33	.90000E+00	.10000E+01	.27298E-05	11037E-04	.00000E+00	10000E+01
34	.70000E+00	.10000E+01	.31621E-05	90767E-05	.00000E+00	10000E+01
35	.50000E+00	.10000E+01	.31999E-05	67858E-05	.00000E+00	10000E+01
36	.30000E+00	.10000E+01	.27364E-05	42788E-05	.00000E+00	10000E+01
37	.10000E+00	.10000E+01	.15265E-05	~.16042E-05	.00000E+00	10000E+01
38	.00000E+00	.92857E+00	.00000E+00	.00000E+00	22960E+01	.39736E+00
39	.00000E+00	.78571E+00	.00000E+00	.00000E+00	57485E+00	.34953E+00
40	.00000E+00	.64286E+00	.00000E+00	.00000E+00	13260E+00	.34608E+00
41	.00000E+00	.50000E+00	.00000E+00	.00000E+00	.19883E+00	.29576E+00
42	.00000E+00	.35714E+00	.00000E+00	.00000E+00	.51615 E +00	.19079E+00

43	.00000E+00	.21429E+00	.00000E+00	.00000E+00	.91311E+00	.30483E-01
44	.00000E+00	.71429E-01	.00000E+00	.00000E+00	.20636E+01	54870E+00

DISPLACEMENTS AT INTERNAL POINTS

POINT	x	Y	υ	v
1	.15000E+01	.25000E+00	27049E-21	11965E-04
2	.15000E+01	.50000E+00	30440E-21	12522E-04
3	.15000E+01	.75000E+00	48638E-21	13175E-04

STRESSES AT THE BOUNDARY NODAL POINTS

NODE	x	Y	SXB	SYB	SXYB
1	.10000E+00	.00000E+00	13018E+01	.00000E+00	.00000E+00
2	.30000E+00	.00000E+00	97418E+00	.00000E+00	.00000E+00
3	.50000E+00	.00000E+00	35074E+00	.00000E+00	.00000E+00
4	.70000E+00	.00000E+00	.16019E+00	.00000E+00	.00000E+00
5	.90000E+00	.00000E+00	.55613E+00	.00000E+00	.00000E+00
6	.11000E+01	.00000E+00	.83728E+00	.00000E+00	.00000E+00
7	.13000E+01	.00000E+00	.10050E+01	.00000E+00	.00000E+00
8	.15000E+01	.00000E+00	.10607E+01	.00000E+00	.00000E+00
9	.17000E+01	.00000E+00	.10050E+01	.00000E+00	.00000E+00
10	.19000E+01	.00000E+00	.83728E+00	.00000E+00	.00000E+00
11	.21000E+01	.00000E+00	.55613E+00	.00000E+00	.00000E+00
12	.23000E+01	.00000E+00	.16019E+00	.00000E+00	.00000E+00
13	.25000E+01	.00000E+00	35074E+00	.00000E+00	.00000E+00
14	.27000E+01	.00000E+00	97418E+00	.00000E+00	.00000E+00
15	.29000E+01	.00000E+00	13018E+01	.00000E+00	.00000E+00
16	.30000E+01	.71429E-01	20636E+01	16270E+00	54870E+00
17	.30000E+01	.21429E+00	91311E+00	18262E+00	.30483E-01
18	.30000E+01	.35714E+00	51615E+00	10323E+00	.19079E+00
19	.30000E+01	.50000E+00	19883E+00	39766E-01	.29576E+00
20	.30000E+01	.64286E+00	.13260E+00	.26520E-01	.34608E+00
21	.30000E+01	.78571E+00	.57485E+00	.11497E+00	.34953E+00
22	.30000E+01	.92857E+00	.22960E+01	39150E+00	.39736E+00
23	.29000E+01	.10000E+01	.13174E+01	10000E+01	.00000E+00
24	.27000E+01	.10000E+01	.63672E+00	10000E+01	.00000E+00
25	.25000E+01	.10000E+01	.12837E-01	10000E+01	.00000E+00
26	.23000E+01	.10000E+01	43507E+00	10000E+01	.00000E+00
27	.21000E+01	.10000E+01	78429E+00	10000E+01	.00000E+00
28	.19000E+01	.10000E+01	10401E+01	10000E+01	.00000E+00
29	.17000E+01	.10000E+01	11968E+01	10000E+01	.00000E+00
30	.15000E+01	.10000E+01	12495E+01	10000E+01	.00000E+00
31	.13000E+01	.10000E+01	11968E+01	10000E+01	.00000E+00
32	.11000E+01	.10000E+01	10401E+01	10000E+01	.00000E+00
33	.90000E+00	.10000E+01	78429E+00	10000E+01	.00000E+00
34	.70000E+00	.10000E+01	43507E+00	10000E+01	.00000E+00
35	.50000E+00	.10000E+01	.12837E-01	10000E+01	.00000E+00
36	.30000E+00	.10000E+01	.63672E+00	10000E+01	.00000E+00
37	.10000E+00	.10000E+01	.13174E+01	10000E+01	.00000E+00
38	.00000E+00	.92857E+00	.22960E+01	39150E+00	39736E+00
39	.00000E+00	.78571E+00	.57485E+00	.11497E+00	34953E+00
40	.00000E+00	.64286E+00	.13260E+00	.26520E-01	34608E+00
41	.00000E+00	.50000E+00	19883E+00	39766E-01	29576E+00
42	.00000E+00	.35714E+00	51615E+00	10323E+00	19079E+00
43	.00000E+00	.21429E+00	91311E+00	18262E+00	30483E-01
44	.00000E+00	.71429E-01	20636E+01	16270E+00	.54870E+00

STRESSES AT THE INTERNAL POINTS

272	BOUNDARY	ELEMENTS

NODE	x	ч	SXIN	SYIN	SXYIN
1	.15000E+01	.25000E+00	.39351E+00	33746E+00	26455E-16
2	.15000E+01	.50000E+00	92069E-01	50317E+00	22378E-15
3	.15000E+01	.75000E+00	58061E+00	66592E+00	54123E-15
*****	******	*********	******	***********	*****

Table 7.1 presents the computed values of the displacement v and the stress σ_x at the cross-section x = 1.5 for various values N. They are compared with those obtained using FEM and the Timoshenko beam theory (i.e. including shear deformation). The deflection at point 2 is given by the expression

$$v_2 = \frac{q\alpha^4}{384EI} \left[1 + 8\kappa (1+\nu) \left(\frac{b}{a}\right)^2 \right]$$

where κ is the shape coefficient of the cross-section ($\kappa = 1.2$ for a rectangular cross-section) and q is the load per unit length ($q = -h t_y$).

Table 7.1	Computed values at boundary and internal points of Example 7.2
	for various values of N .

	BEM						FEM	Beam Theory
Point	Number of boundary elements, N							
	44	244	404	532	804	1604		
Displacement $v \times 10^4$								
A	-0.1381	-0.2633	-0.2766	-0.2815	-0.2864	-0.2911	-0.2924	
2	-0.1252	-0.2495	-0.2627	-0.2675	-0.2724	-0.2771	-0.2783	-0.2885
B	-0.1140	-0.2392	-0.2525	-0.2574	-0.2623	-0.2670	-0.2682	
Stress σ_x								
A	-1.250	-2.491	-2.614	-2.660	-2.706	-2.750	-2.630	-2.250
3	-0.581	-1.055	-1.106	-1.124	-1.143	-1.161	-1.170	-1.125
2	-0.092	-0.090	-0.090	-0.090	-0.090	-0.090	-0.089	0.000
1	0.394	0.872	0.923	0.961	0.961	0.979	0.986	1.125
B	1.061	2.259	2.383	2.475	2.475	2.519	2.450	2.250

For the beam theory, the stress σ_x is computed from the expression

$$\sigma_x = \frac{M}{I} \left(0.5 - y \right)$$

where $M = qa^2/24$ is the bending moment at the mid-section.

We note that a number of 44 constant boundary elements is not sufficient for an accurate solution. The convergence of the BEM was achieved with N=1604 (NX=501, NY=301), whereas of the FEM with 432 rectangular hybrid elements. Finally, the distribution of σ_x at the cross-section x = 3.0 is shown in Fig. 7.10 as it was obtained using BEM, FEM and the Timoshenko beam theory.



Figure 7.10 Distribution of σ_x at the cross-section x = 3.0.

Example 7.3

Determine the deformation and state of stress for the pipe of Fig. 7.11, which is subjected to internal pressure p = 1 MPa. It has a uniform cross-section and since it is very long in the *z*-direction, the produced state of stress is plane strain. The material constants are $E = 2 \times 10^5 \text{ kN/m}^2$ and $\nu = 0.20$. All the other data can be found in Fig. (7.11).

The results are obtained using program ELBECON with INPLANE=0, N=348, NB=2, NL(1)=284 and NL(2)=348. The external boundary is divided into N=142 (NX=91, NY=51) constant boundary elements, while the inner boundary is discretized into N=32 (NX=21, NY=11) elements. The data file is created using program RECTEL-MU.FOR. Due to space considerations, the obtained results are given only at selected points. The distribution of t_y along the boundary y = 0 is shown in Fig. 7.12, whereas the distribution of σ_x along the boundaries y = 0 and y = 1.5 are shown in Fig. 7.13. Fig. 7.14 depicts the deformed shape of the pipe's cross-section. Finally, the stress contours σ_x , σ_y and τ_{xy} at the cross-section of the pipe are presented in Figs. 7.15, 7.16 and 7.17, respectively.


Figure 7.11 Pipe under uniform pressure.

EXAMPLE 7.3 (RESULTS)

.......... J.T. KATSIKADELIS EXAMPLE 7.3 DATA NUMBER OF BOUNDARY ELEMENTS =348 NUMBER OF INTERNAL POINTS = 28 NUMBER OF BOUNDARIES = 2 PLANE STRAIN PROBLEM ELASTIC CONSTANTS: ELASTIC MODULUS = .2000E+06 POISSON RATIO = .20 RESULTS BOUNDARY NODES NODE Y U v TXB TYB х .00000E+00 .00000E+00 -.52996E+03 1 .13736E-01 .00000E+00 -.24189E-04 21 .56319E+00 .00000E+00 -.72425E-03 .00000E+00 .00000E+00 -.35881E+03 41 .11126E+01 .00000E+00 -.35833E-03 .00000E+00 .00000E+00 .84725E+03

46	.12500E+01	.00000E+00	82078E-17	.00000E+00	.00000E+00	.92400E+03
51	.13874E+01	.00000E+00	.35833E-03	.00000E+00	.00000E+00	.84725E+03
71	.19368E+01	.00000E+00	.72425E-03	.00000E+00	.00000E+00	35881E+03
91	.24863E+01	.00000E+00	.24189E-04	.00000E+00	.00000E+00	52996E+03
92	.25000E+01	.14706E-01	.00000E+00	.41985E-04	13765E+03	.00000E+00
112	.25000E+01	.60294E+00	.00000E+00	.14047E-02	24658E+02	.00000E+00
117	.25000E+01	.75000E+00	.00000E+00	.16518E-02	.93068E+01	.00000E+00
122	.25000E+01	.89706E+00	.00000E+00	.18227E-02	.19794E+02	.00000E+00
142	.25000E+01	.14853E+01	.00000E+00	.21831E-02	97037E+03	.00000E+00
143	.24863E+01	.15000E+01	.69463E-04	.22128E-02	.00000E+00	.00000E+00
163	.19368E+01	.15000E+01	.28213E-02	.46238E-02	.00000E+00	.00000E+00
183	.13874E+01	.15000E+01	.15391E-02	.12138E-01	.00000E+00	.00000E+00
188	.12500E+01	.15000E+01	32385E-17	.12683E-01	.00000E+00	.00000E+00
193	.11126E+01	.15000E+01	15391E-02	.12138E-01	.00000E+00	.00000E+00
213	.56319E+00	.15000E+01	28213E-02	.46238E-02	.00000E+00	.00000E+00
233	.13736E-01	.15000E+01	69463E-04	.22128E-02	.00000E+00	.00000E+00
234	.00000E+00	.14853E+01	.00000E+00	.21831E-02	.97037E+03	.00000E+00
254	.00000E+00	.89706E+00	.00000E+00	.18227E-02	19794E+02	.00000E+00
259	.00000E+00	.75000E+00	.00000E+00	.16518E-02	93068E+01	.00000E+00
264	.00000E+00	.60294E+00	.00000E+00	.14047E-02	.24658E+02	.00000E+00
284	.00000E+00	.14706E-01	.00000E+00	.41985E-04	.13765E+03	.00000E+00
285	.75000E+00	.52273E+00	10789E-02	.13739E-02	10000E+04	.00000E+00
290	.75000E+00	.75000E+00	17573E-02	.24620E-02	10000E+04	.00000E+00
295	.75000E+00	.97727E+00	14720E-03	.40572E-02	10000E+04	.00000E+00
296	.77381E+00	.10000E+01	.18298E-02	.70657E-02	.00000E+00	.10000E+04
304	.11548E+01	.10000E+01	.69300E-03	.13851E-01	.00000E+00	.10000E+04
306	.12500E+01	.10000E+01	.68200E-18	.14123E-01	.00000E+00	.10000E+04
308	.13452E+01	.10000E+01	69300E-03	.13851E-01	.00000E+00	.10000E+04
316	.17262E+01	.10000E+01	18298E-02	.70657E-02	.00000E+00	.10000E+04
317	.17500E+01	.97727E+00	.14720E-03	.40572E-02	.10000E+04	.00000E+00
322	.17500E+01	.75000E+00	.17573E-02	.24620E-02	.10000E+04	.00000E+00
327	.17500E+01	.52273E+00	.10789E-02	.13739E-02	.10000E+04	.00000E+00
328	.17262E+01	.50000E+00	52574E-04	21925E-03	.00000E+00	10000E+04
336	.13452E+01	.50000E+00	15442E-04	23685E-02	.00000E+00	10000E+04
338	.12500E+01	.50000E+00	.44900E-18	24171E-02	.00000E+00	10000E+04
340	.11548E+01	.50000E+00	.15442E-04	23685E-02	.00000E+00	10000E+04
348	.77381E+00	.50000E+00	.52574E-04	21925E-03	.00000E+00	10000E+04

DISPLACEMENTS AT INTERNAL POINTS

POINT	x	Y	U	v
1	.50000E-01	.75000E+00	17973E-04	.16589E-02
3	.25000E+00	.75000E+00	11601E-03	.18458E-02
5	.45000E+00	.75000E+00	39619E-03	.22240E-02
8	.65000E+00	.75000E+00	11736E-02	.25230E-02
9	.70000E+00	.75000E+00	14580E-02	.25163E-02
10	.18000E+01	.75000E+00	.14580E-02	.25163E-02
11	.18500E+01	.75000E+00	.11736E-02	.25230E-02
14	.20500E+01	.75000E+00	.39619E-03	.22240E-02
16	.22500E+01	.75000E+00	.11601E-03	.18458E-02
18	.24500E+01	.75000E+00	.17973E-04	.16589E-02
19	.12500E+01	.50000E-01	29477E-18	24514E-03
21	.12500E+01	.25000E+00	47095E-18	12253E-02
23	.12500E+01	.45000E+00	34305E-18	21906E-02
24	.12500E+01	.10500E+01	85212E-18	.13990E-01
26	.12500E+01	.12500E+01	13705E-17	.13330E-01
28	.12500E+01	.14500E+01	22311E-17	.12828E-01

STRESSES AT THE BOUNDARY NODAL POINTS

NODE	х	Y	SXB	SYB	SXYB

1	.13736E-01	.00000E+00	76833E+02	.52996E+03	.00000E+00
21	.56319E+00	.00000E+00	14872E+03	.35881E+03	.00000E+00
41	.11126E+01	.00000E+00	.30248E+03	84725E+03	.00000E+00
46	.12500E+01	.00000E+00	.32588E+03	92400E+03	.00000E+00
51	.13874E+01	.00000E+00	.30248E+03	84725E+03	.00000E+00
71	.19368E+01	.00000E+00	14872E+03	.35881E+03	.00000E+00
91	.24863E+01	.00000E+00	76833E+02	.52996E+03	.00000E+00
92	.25000E+01	.14706E-01	13765E+03	.38094E+03	.00000E+00
112	.25000E+01	.60294E+00	24658E+02	.38745E+03	.00000E+00
117	.25000E+01	.75000E+00	.93068E+01	.30245E+03	.00000E+00
122	.25000E+01	.89706E+00	.19794E+02	.18846E+03	.00000E+00
142	.25000E+01	.14853E+01	97037E+03	24593E+02	.00000E+00
143	.24863E+01	.15000E+01	74157E+03	28422E-13	.00000E+00
163	.19368E+01	.15000E+01	98134E+03	28422E-13	.00000E+00
183	.13874E+01	.15000E+01	.21656E+04	.00000E+00	.00000E+00
188	.12500E+01	.15000E+01	.24140E+04	.56843E-13	.00000E+00
193	.11126E+01	.15000E+01	.21656E+04	.56843E-13	.00000E+00
213	.56319E+00	.15000E+01	98134E+03	28422E-13	.00000E+00
233	.13736E-01	.15000E+01	74157E+03	28422E-13	.00000E+00
234	.00000E+00	.14853E+01	97037E+03	24593E+02	.00000E+00
254	.00000E+00	.89706E+00	.19794E+02	.18846E+03	.00000E+00
259	.00000E+00	.75000E+00	.93068E+01	.30245E+03	.00000E+00
264	.00000E+00	.60294E+00	24658E+02	.38745E+03	.00000E+00
284	.00000E+00	.14706E-01	13765E+03	.38094E+03	.00000E+00
285	.75000E+00	.52273E+00	10000E+04	.38357E+04	.00000E+00
290	.75000E+00	.75000E+00	10000E+04	.76988E+03	.00000E+00
295	.75000E+00	.97727E+00	10000E+04	.74041E+04	.00000E+00
296	.77381E+00	.10000E+01	.45463E+04	10000E+04	.00000E+00
304	.11548E+01	.10000E+01	17027E+04	10000E+04	.00000E+00
306	.12500E+01	.10000E+01	17871E+04	10000E+04	.00000E+00
308	.13452E+01	.10000E+01	17027E+04	10000E+04	.00000E+00
316	.17262E+01	.10000E+01	.45463E+04	10000E+04	.00000E+00
317	.17500E+01	.97727E+00	10000E+04	.74041E+04	.00000E+00
322	.17500E+01	.75000E+00	10000E+04	.76988E+03	.00000E+00
327	.17500E+01	.52273E+00	10000E+04	.38357E+04	.00000E+00
328	.17262E+01	.50000E+00	.24004E+04	10000E+04	.00000E+00
336	.13452E+01	.50000E+00	29043E+03	10000E+04	.00000E+00
338	.12500E+01	.50000E+00	28146E+03	10000E+04	.00000E+00
340	.11548E+01	.50000E+00	29043E+03	10000E+04	.00000E+00
348	.77381E+00	.50000E+00	.24004E+04	10000E+04	.00000E+00

STRESSES AT THE INTERNAL POINTS

NODE	x	Y	SXIN	SYIN	SXYIN
1	.50000E-01	.75000E+00	.71437E+01	.30748E+03	.33599E+02
3	.25000E+00	.75000E+00	50055E+02	.43360E+03	.17188E+03
5	.45000E+00	.75000E+00	28878E+03	.78449E+03	.27850E+03
8	.65000E+00	.75000E+00	85555E+03	.11362E+04	.18253E+03
9	.70000E+00	.75000E+00	96161E+03	.10218E+04	.97657E+02
10	.18000E+01	.75000E+00	96161E+03	.10218E+04	97657E+02
11	.18500E+01	.75000E+00	85555E+03	.11362E+04	18253E+03
14	.20500E+01	.75000E+00	28878E+03	.78449E+03	27850E+03
16	.22500E+01	.75000E+00	50055E+02	.43360E+03	17188E+03
18	.24500E+01	.75000E+00	.71437E+01	.30748E+03	33599E+02
19	.12500E+01	.50000E-01	.32130E+03	92963E+03	45669E-12
21	.12500E+01	.25000E+00	.19696E+03	98046E+03	48556E-12
23	.12500E+01	.45000E+00	17085E+03	10010E+04	11087E-11
24	.12500E+01	.10500E+01	12449E+04	95462E+03	.34186E-11
26	.12500E+01	.12500E+01	.38464E+03	52482E+03	.10554E-11
28	.12500E+01	.14500E+01	.18800E+04	39172E+02	20633E-11
*****	******	*****	******	*****	****



Figure 7.12 Distribution of t_y along the boundary y = 0.



Figure 7.13 Distribution of σ_x along the boundaries y = 0 and y = 1.5.



Figure 7.14 Deformed pipe cross-section.



Figure 7.15 Contours of σ_x at the pipe cross-section.



Figure 7.16 Contours of σ_y at the pipe cross-section.



Figure 7.17 Contours of τ_{xy} at the pipe cross-section.

7.13 References

The boundary value problems of the theory of elasticity are well known and the interested reader find them in many relevant books, some of which are listed below. References [1, 2, 3, 11, 12, 13] are addressed mainly to engineers, while

[14, 15, 16] are more theoretical. Regarding the application of BEM to two- or three-dimensional elastostatic problems, the reader is advised to consult references [6] and [17–26]. The BEM has also been employed to analyze composite plane bodies with perfect [27] or unilateral [28] bonding at the interfaces. Recently, the two-dimensional elastostatic problem was treated by the analog equation method [29], a BEM-based solution method that converts Navier's equations into two uncoupled Poisson's equations simplifying, therefore, significantly the solution.

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Problems

7.1. Derive the boundary integral equations for the plain strain problem taking into account initial strains.

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- 7.2. Determine the deformation and the stresses in the pipe of Fig. 7.11, which are produced by a temperature change of $\Delta T = 80^{\circ}$ C. The coefficient of thermal dilatation is $\alpha = 10^{-5}$, while the elastic constants are those of Example 7.3.
- 7.3. Determine the deformation and the stresses in the pipe of Fig. 7.11, when the temperature of the fluid in the pipe is $T_{in} = 300^{\circ}$ C, the ambient temperature is $T_{out} = 25^{\circ}$ C, the reference temperature (temperature of manufacturing for the pipe) is $T_0 = 10^{\circ}$ C and the internal pressure is p = 1 MPa. The coefficient of thermal dilatation is $\alpha = 10^{-5}$.
- 7.4. Determine the deformation and the stresses in the deep beam of Fig. 7.9 considering also the weight of the body, $\rho_w = 25 \text{ kN/m}^3$.
- **7.5.** The two-column frame is subjected to uniformly distributed horizontal loads p = 750 kN/m as shown in Fig. P7.5. The supporting wall is fixed (u = 0 and v = 0) along its sides AB, BC, CD, DE and EF. Determine the deformed shape of the frame and evaluate its lateral stiffness K_{cl} using the theory of elasticity. The thickness of the structure is 0.25 m and the elastic constants are $E = 2.1 \times 10^7 \text{ kN/m}^2$ and $\nu = 0.20$.





7.6. The shear wall of the following figure is subjected to distributed horizontal loads p = 50 kN/m. Determine the deformation of the plane body and find

the stress distribution along the cross-sections x = 0.00, 0.90, 1.70 m and y = 0.40, 1.60, 4.00 m. The thickness is 0.25 m and the elastic constants are $E = 2.1 \times 10^7 \text{ kN/m}^2$ and $\nu = 0.15$. The support conditions of the structure are u(x, 0) = 0 and v(x, 0) = 0.



Figure P7.6

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Derivatives of r

In this Appendix, we present certain relations which facilitate the differentiation of the kernels of the integral equations. Points inside the domain Ω are denoted by upper case letters, e.g. P(x, y), while points on the boundary by lower case letters, e.g. $q(\xi, \eta)$. The angle between the x-axis and the vector r is denoted by α and the angle between the x-axis and the unit vector \mathbf{n} normal to the boundary at point q by β (see Fig. A.1). Using these two angles, we also define the angle ϕ as

$$\phi = \text{angle}(\mathbf{r}, \mathbf{n}) = \beta - \alpha \tag{A.1}$$

Referring to Fig. A.1, it is

$$\cos \alpha = \frac{\xi - x}{r} \tag{A.2}$$

$$\sin \alpha = \frac{\eta - y}{r} \tag{A.3}$$

where

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$
(A.4)

Differentiation of Eq. (A.4) yields

$$r_{x} = -r_{\xi} = -\frac{\xi - x}{r} = -\cos\alpha$$
 (A.5)

$$r_y = -r_\eta = -\frac{\eta - y}{r} = -\sin\alpha \tag{A.6}$$

where the subscripts x, y, ξ and η preceded by a comma denote differentiation with respect to the corresponding independent variable.



Figure A.1 Geometric definitions related to the relative position of a field point P and a boundary point q.

Noting that

 $\cos\beta = n_x$

 $\sin\beta=n_y$

we can derive the following expressions for the derivatives of r

$$r_{,n} = r_{,\xi} n_{x} + r_{,\eta} n_{y}$$

$$= r_{,\xi} \cos \beta + r_{,\eta} \sin \beta$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$= \cos (\beta - \alpha)$$

$$= \cos \phi$$

$$r_{,t} = -r_{,\xi} n_{y} + r_{,\eta} n_{x}$$

$$= -r_{,\xi} \sin \beta + r_{,\eta} \cos \beta$$

$$= -\cos \alpha \sin \beta + \sin \alpha \cos \beta$$

$$= -\sin (\beta - \alpha)$$

$$= -\sin \phi$$
(A.8)

$$r_{,xx} = -\left(\frac{\xi - x}{r}\right)_{,x}$$

= $-\frac{(\xi - x)_{,x}r - (\xi - x)r_{,x}}{r^2}$
= $\frac{(\eta - y)^2}{r^3}$
= $\frac{r_y^2}{r}$ (A.9)

Similarly, we obtain

$$r_{yy} = \frac{r_x^2}{r} \tag{A.10}$$

$$r_{xy} = -\frac{r_x r_y}{r} \tag{A.11}$$

$$r_{\xi\xi} = \frac{r_{\eta}^2}{r} \tag{A.12}$$

$$r_{,\eta\eta} = \frac{r_{\xi}^2}{r} \tag{A.13}$$

$$r_{\xi\eta} = -\frac{r_{\xi}r_{,\eta}}{r} \tag{A.14}$$

$$r_{\xi x} = -r_{xx} = -\frac{r_{y}^{2}}{r}$$
(A.15)

$$r_{\xi y} = -r_{xy} = \frac{r_{x}r_{y}}{r}$$
 (A.16)

$$r_{,\eta x} = -r_{,yx} = \frac{r_{,x}r_{,y}}{r}$$
 (A.17)

$$r_{,\eta y} = -r_{,yy} = -\frac{r_{,x}^2}{r}$$
 (A.18)

$$r_{,nx} = (r_{,\xi} \cos \beta + r_{,\eta} \sin \beta)_{,x}$$
$$= r_{,\xi x} \cos \beta + r_{,\eta x} \sin \beta =$$

$$= \frac{r_y}{r} (r_\eta \cos\beta - r_\xi \sin\beta)$$

$$= \frac{r_t r_y}{r}$$
(A.19)
$$r_{ny} = (r_\xi \cos\beta + r_\eta \sin\beta)_{,y}$$

$$= r_{,\xi y} \cos\beta + r_{,\eta y} \sin\beta$$

$$= \frac{r_x}{r} (-r_{,\eta} \cos\beta + r_{,\xi} \sin\beta)$$

$$= -\frac{r_t r_x}{r}$$
(A.20)
$$r_{,tx} = (-r_\xi \sin\beta + r_{,\eta} \cos\beta)_{,x}$$

$$= -r_{,\xi x} \sin\beta + r_{,\eta x} \cos\beta$$

$$= -\frac{r_y}{r} (r_\xi \cos\beta + r_{,\eta} \sin\beta)$$

$$= -\frac{r_y r_{,ty}}{r}$$
(A.21)
$$r_{,ty} = (-r_{,\xi} \sin\beta + r_{,\eta} \cos\beta)_{,y}$$

$$= -r_{,\xi \eta} \sin\beta + r_{,\eta y} \cos\beta$$

$$= \frac{r_x}{r} (r_{,\xi} \cos\beta + r_{,\eta} \sin\beta)$$

$$=\frac{r_{x}r_{n}}{r} \tag{A.22}$$

Gauss Integration

B.1 Gauss integration of a regular function

The success of BEM as a computational method is contingent upon evaluating accurately line and domain integrals. Any numerical integration method can be utilized, e.g. trapezoidal rule, Simpson rule, Newton-Cotes rule, etc. The numerical methods approximate an integral with a sum of products of the values of the integrand at specific points of the integration interval, usually equidistant, multiplied by known coefficients, the *weight factors* (or *weights*), resulting from the employed integration rule. Hence, an essential criterion for choosing the integration rule is the achievement of the desired degree of accuracy in the approximation of the integral by utilizing the lowest possible number of integration points. The Gauss integration method satisfies this criterion. In this method, the points are not equidistant but they arrange themselves in the optimal pattern. The weight factors are also adjusted in this optimization process. The concept behind the Gauss integration is quite simple, and will be explained in the sequel.

Let us consider the integral

$$I = \int_{-1}^{1} f(\xi) d\xi$$
 (B.1)

It is approximated by the sum

$$I \approx I_n = \sum_{k=1}^n f(\xi_k) w_k \tag{B.2}$$

where $f(\xi_k)$ (k = 1, 2, ..., n) are the values of the function $f(\xi)$ at the *n* points ξ_k $(-1 < \xi_k < +1)$ and w_k the associated *weights*. The points ξ_k , which are called *Gauss integration points* or simply *Gauss points*, are not equidistant within the integration interval, but, for a given *n*, their location and the associated weights are chosen so that the error $E_n = I - I_n$ becomes minimum.

We first study the trivial case where the function $f(\xi)$ is a linear polynomial

$$f(\xi) = a_0 + a_1 \, \xi \tag{B.3}$$

The exact value of the integral is

$$I = \int_{-1}^{+1} (a_0 + a_1 \xi) d\xi$$
$$= \left[a_0 \xi + a_1 \frac{\xi^2}{2} \right]_{-1}^{+1}$$
$$= 2a_0$$

while the approximate value obtained from Eq. (B.2) for n = 1 is

$$I_1 = (a_0 + a_1 \xi_1) w_1$$

Therefore, the error is

$$E_1(a_0, a_1) = I - I_1 = 2a_0 - (a_0 + a_1\xi_1)w_1$$

and is minimized for

$$\frac{\partial E_1}{\partial a_0} = 2 - w_1 = 0$$

$$\frac{\partial E_1}{\partial a_1} = -w_1 \xi_1 = 0$$
(B.4)

which yield

 $\xi_1 = 0$, $w_1 = 2$

and $E_1(a_0, a_1) = 0$.

We notice that Eq. (B.2) gives

$$\int_{-1}^{1} f(\xi) d\xi = 2f_0$$

= 2a_0 (B.5)

where $f_0 = f(0)$.

Namely, the value of the integral is independent of the slope a_1 of the straight line representing the function. Hence, the definite integrals on the interval [-1,+1] of all the linear functions passing through the point $(0, f_0)$ have the same value (see Fig. B.1).



Figure B.1 Typical linear polynomials.

Consider now the case of a function $f(\xi)$ which is a quadratic polynomial. The exact value of the integral is

$$I = \int_{-1}^{1} \left(a_0 + a_1 \xi + a_2 \xi^2 \right) d\xi$$
$$= \left[a_0 \xi + a_1 \frac{\xi^2}{2} + a_2 \frac{\xi^3}{3} \right]_{-1}^{+1}$$
$$= 2a_0 + \frac{2}{3}a_2$$

while the approximate one for two Gauss points (n=2) is

$$I_{2} = f(\xi_{1})w_{1} + f(\xi_{2})w_{2}$$

= $\left(a_{0} + a_{1}\xi_{1} + a_{2}\xi_{1}^{2}\right)w_{1} + \left(a_{0} + a_{1}\xi_{2} + a_{2}\xi_{2}^{2}\right)w_{2}$

The resulting error is

$$E_{2}(a_{0}, a_{1}, a_{2}) = I - I_{2}$$

$$= 2a_{0} + \frac{2}{3}a_{2} - \left(a_{0} + a_{1}\xi_{1} + a_{2}\xi_{1}^{2}\right)w_{1}$$

$$- \left(a_{0} + a_{1}\xi_{2} + a_{2}\xi_{2}^{2}\right)w_{2}$$
(B.6)

which is minimized under the following conditions

$$\frac{\partial E_2}{\partial a_0} = 2 - w_1 - w_2 = 0 \tag{B.7}$$

$$\frac{\partial E_2}{\partial a_1} = -\xi_1 \, w_1 - \xi_2 \, w_2 = 0 \tag{B.8}$$

$$\frac{\partial E_2}{\partial a_2} = \frac{2}{3} - \xi_1^2 w_1 - \xi_2^2 w_2 = 0 \tag{B.9}$$

The above three simultaneous equations include four unknowns, w_1 , w_2 , ξ_1 and ξ_2 . Therefore, one of them can be chosen arbitrarily, and since it is convenient to place the points ξ_1 and ξ_2 symmetrically with respect to the origin, we set

$$\xi_2 = -\xi_1 \tag{B.10}$$

From Eq. (B.8), we obtain $w_1 = w_2$ and from Eq. (B.7) we find

$$w_1 = w_2 = 1$$
 (B.11)

Finally, Eq. (B.9) gives

$$\xi_1 = \frac{1}{\sqrt{3}} = 0.577350269189626 \tag{B.12}$$

Introducing the above values of ξ_1 , ξ_2 , w_1 and w_2 into Eq. (B.6), the error becomes $E_2(a_0, a_1, a_2) = 0$. The approximate value of the integral is computed from the sum

$$I_2 = f(\xi_1) w_1 + f(\xi_2) w_2 \tag{B.13}$$

where ξ_1 and ξ_2 are given by Eqs. (B.10) and (B.12) and w_1 , w_2 by Eq. (B.11).

We consider the case where the function $f(\xi)$ is a cubic polynomial and we apply the same procedure. The exact value is

$$egin{aligned} I_3 &= \int_{-1}^1 \Bigl(a_0 + a_1\,\xi + a_2\,\xi^2 + a_3\,\xi^3 \, \Bigr) d\xi \ &= 2a_0 + rac{2}{3}a_2 \end{aligned}$$

Using again two Gauss points (n = 2), the expression for the error becomes

$$E_{3}(a_{0}, a_{1}, a_{2}, a_{3}) = \left(2a_{0} + \frac{2}{3}a_{2}\right) - \left(a_{0} + a_{1}\xi_{1} + a_{2}\xi_{1}^{2} + a_{3}\xi_{1}^{3}\right)w_{1}$$
$$- \left(a_{0} + a_{1}\xi_{2} + a_{2}\xi_{2}^{2} + a_{3}\xi_{2}^{3}\right)w_{2}$$

which is minimized if the following conditions are satisfied

$$\frac{\partial E_3}{\partial a_0} = 2 - w_1 - w_2 = 0 \tag{B.14a}$$

$$\frac{\partial E_3}{\partial a_1} = -\xi_1 \, w_1 - \xi_2 \, w_2 = 0 \tag{B.14b}$$

$$\frac{\partial E_3}{\partial a_2} = \frac{2}{3} - \xi_1^2 w_1 - \xi_2^2 w_2 = 0$$
(B.14c)

$$\frac{\partial E_3}{\partial a_3} = -\xi_1^3 w_1 - \xi_2^3 w_2 = 0 \tag{B.14d}$$

The solution $w_1 = w_2 = 1$ and $\xi_1 = -\xi_2 = 1/\sqrt{3}$, obtained when $f(\xi)$ is a second order polynomial, is also a solution of the simultaneous equations (B.14), and yields $E_3(a_0, a_1, a_2, a_3) = 0$. Hence, Eq. (B.2) renders exact values for the integrals of the second and third order polynomials using the same Gauss integration rule, i.e. the same integration points and weights.

This procedure can also be applied for higher order polynomials. In general, it can be proved that the formula of Eq. (B.2) with n terms gives the exact value for the integral of a polynomial $f(\xi)$ of order less than or equal to 2n - 1. However, this procedure can hardly be employed to determine the coordinates of the Gauss points and the weight factors, because it becomes complicated and laborious for increasing degree of the polynomial. The Gauss points can be obtained by a simpler method, which uses Legendre's orthogonal polynomials [1, 2] to represent the function $f(\xi)$ and that is why it is known as *Gauss-Legendre integration*.

Legendre's polynomials are defined as

$$P_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n$$
(B.15)

For example, the first, second and third order polynomials are

for
$$n = 1$$
: $P_1(\xi) = \xi$
for $n = 2$: $P_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$
for $n = 3$: $P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$

(B.16)

It is proven that the coordinates ξ_k of the integration points are the zeros of these polynomials. The weight factors w_k are computed by the expression

$$w_{k} = \frac{2(1-\xi_{k}^{2})}{n^{2} \left[P_{n-1}(\xi_{k})\right]^{2}}$$
(B.17)

Table B.1 provides the coordinates of the integration points and the weights for various values of n.

If $f(\xi)$ is not a polynomial, the integral can be evaluated approximately as

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{k=1}^{n} f(\xi_k) w_k$$
(B.18)

where the function is actually approximated by a polynomial of degree 2n-1. The error associated with the Gauss-Legendre integration is given by Lanczos in the following form [1]

$$E_n = I - I_n = \frac{1}{2n+1} \left[f(1) + f(-1) - I_n - \sum_{k=1}^n w_k \,\xi_k \, f'(\xi_k) \right] \tag{B.19}$$

in which n is the number of integration points. This estimation is very good for smooth functions $f(\xi)$.

Usually, the interval over which a function is integrated differs from the interval [-1,+1]. Suppose that we have to evaluate the integral

$$I = \int_{a}^{b} f(x) dx \tag{B.20}$$

Using the transformation

$$x = \frac{b-a}{2}\xi + \frac{b+a}{2}$$
(B.21)

	$I = \int_{-\infty}^{1} f(\xi) d\xi \approx \sum_{n=1}^{n} f(\xi) d\xi$	$f(\xi_k) w_k$
	J_{-1}	
<i>n</i>	$\pm \xi_k$	w_k
n = 2	0.57735 02691 89626	1.00000 00000 00000
n = 3	0.0000 00000 00000	0.88888 88888 88889
	0.77459 66692 41483	0.55555 55555 55556
$\overline{n} = 4$	0.33998 10435 84856	0.65214 51548 62546
	0.86113 63115 94053	0.34785 48451 37454
n = 5	0.00000 00000 00000	0.56888 88888 88889
	0.53846 93101 05683	0.47862 86704 99366
	0.90617 98459 38664	0.23692 68850 56189
n = 6	0.23861 91860 83197	0.46791 39345 72691
	0.66120 93864 66265	0.36076 15730 48139
	0.93246 95142 03152	0.17132 44923 79170
n = 7	0.0000 00000 00000	0.41795 91836 73469
	0.40584 51513 77397	0.38183 00505 05119
	0.74153 11855 99394	0.27970 53914 89277
	0.94910 79123 42759	0.12948 49661 68870
$\frac{1}{n=8}$	0 18343 46424 95650	0.36268 37833 78362
	0.52553 24099 16329	0.31370 66458 77887
	0 79666 64774 13627	0 22238 10344 53374
	0.96028 98564 97536	0.10122 85362 90376
<u> </u>	0,0000,0000,0000	0 33023 93550 01260
n = 3	0.32425 34234 03809	0.31234 70770 40003
	0.61337 14327 00590	0.26061_06964_02935
	0.83603 11073 26636	0.18064 81606 94857
	0.96816 02395 07626	0.08127 43883 61574
10	0 14997 42290 91(21	0.20552 42247 14752
n = 10	0.14887 43389 81031	0.29332 42247 14733
	0.43339 33941 29247	0.20920 07193 09990
	0.07940 93082 99024	0.21908 03023 13982
	0.80300 33000 88785	0.06667 13443 08688
<u> </u>		
n = 12	0.12523 34085 11469	0.24914 70458 13403
	0.36783 14989 98180	0.23349 25365 38355
	0.58/31 /9542 8061/	0.20310 /420/ 23000
	0.76990 26741 94303	0.10007 83283 43340
	0.90411 /2303 /04/3	0.10093 93239 93318
	0.70130 00342 40/19	0.04/17 55505 60512
n = 16	0.09501 25098 37637	0.18945 06104 55068
	0.28160 35507 79258	0.18260 34150 44923
	0.45801 6/776 57227	0.16915 65193 95002
	0.01/8/ 02444 02643	0.14939 39888 103/0
	0.75540 44083 55003	0.0515 05114 02402
	0.00303 12023 87831	0.07313 03110 02472
	0.98940 09349 91649	0.02715 24594 11754
		0.02/10 2.09/1 11/01

Table B.1 Abscissas and weights for the Gauss-Legendre integration.

n	ξ_k	w_k	$\int_1^3 \frac{\sin^2 x}{x} dx$
1	$\xi_1 = 0$	$w_1 = 2.0000000000$	0.82682181
2	$\xi_1 = -0.5773502692$	$w_1 = 1.0000000000$	0.79856002
	$\xi_2 = -\xi_1$	$w_2 = w_1$	
3	$\xi_1 = -0.7745966692$	$w_1 = 0.5555555556$	0.79465269
	$\xi_2=0$	$w_2 = 0.8888888889$	
	$\xi_3=-\xi_1$	$w_3 = w_1$	
4	$\xi_1 = -0.8611363116$	$w_1 = 0.3478548451$	0.79482835
	$\xi_2 = -0.3399810436$	$w_2 = 0.6521451549$	
	$\xi_3=-\xi_2$	$w_3 = w_2$	
	$\xi_4 = -\xi_1$	$w_4 = w_1$	
5	$\xi_1 = -0.9061798459$	$w_1 = 0.2369268851$	0.79482516
	$\xi_2 = -0.5384693101$	$w_2 = 0.4786286705$	
	$\xi_3=0$	$w_3 = 0.5688888889$	
	$\xi_4=-\xi_2$	$w_4 = w_2$	
	$\xi_5 = -\xi_1$	$w_5 = w_1$	
Exact value			0.79482518

Table B.2 Computed values of the integral $I = \int_{1}^{3} \frac{\sin^{2} x}{x} dx$ for various numbers of Gauss points, n.

the interval of Eq. (B.20) is mapped onto the interval [-1, +1]. Thus,

$$I = \int_{a}^{b} f(x) dx = \frac{b-a}{2} \int_{-1}^{+1} f(\xi) d\xi$$
(B.22)

Example B.1

Determine the value of the integral

$$I = \int_1^3 \frac{\sin^2 x}{x} \, dx$$

The transformation $x = \xi + 2$ yields the expression of the above integral in the interval [-1, +1], which is of the following form

$I = \int_{-1}^{1} f(\xi) d\xi \approx \sum_{k=1}^{n} f(\xi_k) w_k$						
n	ξ_k	w_k				
n=2	0.11200 88061 66976 0.60227 69081 18738	0.71853 93190 30384 0.28146 06809 69615				
n = 3	0.06389 07930 873254 0.36899 70637 15618 0.76688 03039 38941	0.51340 45522 32363 0.39198 00412 01487 0.09461 54065 661491				
n = 4	0.04144 84801 993832 0.24547 49143 20602 0.55616 54535 60278 0.84898 23945 32985	0.38346 40681 45135 0.38687 53177 74762 0.19043 51269 50142 0.03922 54871 299598				
n = 5	0.02913 44721 519720 0.17397 72133 20897 0.41170 25202 84902 0.67731 41745 82820 0.89477 13610 31008	0.29789 34717 82894 0.34977 62265 13224 0.23448 82900 44052 0.09893 04595 166331 0.01891 15521 431957				
n = 6	0.02163 40058 441169 0.12958 33911 54950 0.31402 04499 14765 0.53865 72173 51802 0.75691 53373 77402 0.92266 88513 72120	0.23876 36625 78547 0.30828 65732 73946 0.24531 74265 63210 0.14200 87565 66476 0.05545 46223 248862 0.01016 89586 929322				
n = 7	0.01671 93554 082581 0.10018 56779 15675 0.24629 42462 07930 0.43346 34932 57033 0.63235 09880 47766 0.81111 86267 40105 0.94084 81667 43347	0.19616 93894 25248 0.27030 26442 47272 0.23968 18730 07690 0.16577 57748 10432 0.08894 32271 376579 0.03319 43043 565710 0.05932 78701 512592				
n = 8	0.01332 02441 608924 0.07975 04290 138949 0.19787 10293 26188 0.35415 39943 51909 0.52945 85752 34917 0.70181 45299 39099 0.84937 93204 41106 0.95332 64500 56359	0.16441 66047 28002 0.23752 56100 23306 0.22684 19844 31919 0.17575 40790 06070 0.11292 40302 46759 0.05787 22107 177820 0.02097 90737 421329 0.03686 40710 402761				

Table B.3Abscissas and weights for the Gaussian quadrature of a
function with logarithmic singularity.

$$I = \int_{-1}^{+1} \frac{\sin^2(\xi+2)}{\xi+2} \, d\xi$$

The integral is computed using Gaussian integration. Results for various values of n are presented in Table B.2. We notice that a five-point Gaussian quadrature is sufficient to approximate the value of the integral to seven significant figures.

B.2 Integrals with a logarithmic singularity

The Gaussian integration gives good results when a function $f(\xi)$ varies smoothly within the integration interval. If the integrand is singular the Gauss-Legendre integration cannot be employed and for this purpose special integration rules have been developed which depend on the type of singularity. Integrals whose integrand includes a logarithmic singularity are approximated by the formula [3]

$$\int_0^1 f(\xi) \, \ell n\xi \, d\xi \approx \sum_{k=1}^n f(\xi_k) \, w_k \tag{B.23}$$

The integration points ξ_k and the weights w_k are given in Table B.3.

B.3 Double integrals of a regular function

The Gaussian integration can also be employed to evaluate double integrals. In this case the integration rule depends on the geometry of the domain over which the integration is performed. Various Gauss integration rules have been developed for domains of specific geometry, such as rectangles, triangles, circles as well as domains that can be transformed into one of those geometries. Efforts have been made to develop Gauss integration rules for domains of arbitrary geometry, e.g. the method of finite sectors [4]. In the sequel, we will limit our discussion to Gaussian quadratures for rectangular and triangular domains, which are the common geometries for two-dimensional discretizations.

B.3.1 Gauss integration for rectangular domains

For domains resulting as Cartesian products of lower dimensions, e.g. squares, cubes, cylinders, etc., integration rules can always be formulated by *multiplying* rules of lower dimensions [2, 5]. For example, if

$$\int_{-1}^{+1} f(\xi) d\xi \approx \sum_{k=1}^{n} f(\xi_k) w_k$$
(B.24)

is a one-dimensional rule, then

$$\int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) \, d\xi \, d\eta \approx \sum_{j=1}^{n} \sum_{i=1}^{n} f(\xi_j, \eta_i) \, w_i \, w_j \tag{B.25}$$

Number and position of the Gauss points	x_k	y_k	w_k
	0	0	4⁄9
	$\pm h$	$\pm h$	1/36
	$\pm h$	0	1⁄9
n = 9	0	$\pm h$	$\frac{1}{9}$
• • • • n = 4	$\pm h \frac{\sqrt{3}}{3}$	$\pm h \frac{\sqrt{3}}{3}$	1⁄4
	0	0	16/81
	$\pm h\sqrt{\frac{3}{5}}$	$\pm h\sqrt{\frac{3}{5}}$	25/324
	0	$\pm h\sqrt{\frac{3}{5}}$	¹⁰ / ₈₁
n = 9	$\pm h\sqrt{\frac{3}{5}}$	0	10/81

Table B.4Coordinates and weights for Gauss integration on square
domains of side 2h.

is the corresponding two-dimensional rule. These rules, however, are not necessarily on the side of economy, with regard to the number of integration points. For square domains $\Omega: |x| < h$, |y| < h, the following simple rule can be employed [2]

$$\frac{1}{4h^2} \iint_{\Omega} f(x,y) \, dx \, dy \approx \sum_{k=1}^n f(x_k, y_k) \, w_k \tag{B.26}$$

The coordinates of the Gauss points x_k and y_k and the associated weight factors are given in Table B.4. The order of the error is $E = O(h^4)$ for the first two cases and $E = O(h^6)$ for the third one.

B.3.2 Gauss integration for triangular domains

The integration over a triangular domain is simplified by employing a natural coordinate system, which here is referred to as *triangular coordinate system* (see Fig. B.2). The sides a_1 , a_2 , a_3 are designated by the number of the opposite vertices. The triangular coordinates ξ_i (i = 1, 2, 3) for an interior point P are defined as the ratios of the areas A_i to the total area A of the triangle,

$$\xi_1 = \frac{A_1}{A}, \quad \xi_2 = \frac{A_2}{A}, \quad \xi_3 = \frac{A_3}{A}$$
 (B.27)

Since the sum of the three areas is

$$A_1 + A_2 + A_3 = A$$

it is evident that

$$\xi_1 + \xi_2 + \xi_3 = 1 \tag{B.28}$$



Figure B.2 Triangular coordinate systems.

The Cartesian coordinates x and y of point P are related to the triangular coordinates as

$$x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3$$
 (B.29)

where x_i , y_i (i = 1, 2, 3) are the coordinates of the triangle's vertices. Equations (B.29) can be checked at a few specific points. For example, at the centroid of the triangle it is:

$$A_1 = A_2 = A_3 = rac{A}{3},$$

 $x = rac{x_1 + x_2 + x_3}{3}$ and $y = rac{y_1 + y_2 + y_3}{3}$

while at point 2 (see Fig. B.2): $A_2 = A$, $A_1 = A_3 = 0$, $x = x_2$ and $y = y_2$.

Equations (B.28) and (B.29) yield the relation between rectangular and orthogonal coordinates which can be written in matrix form as

$$\begin{cases} 1\\x\\y \end{cases} = \begin{bmatrix} 1 & 1 & 1\\x_1 & x_2 & x_3\\y_1 & y_2 & y_3 \end{bmatrix} \begin{cases} \xi_1\\\xi_2\\\xi_3 \end{cases}$$
(B.30)



Figure B.3 Integration over a triangular domain.

The integration over the triangular domain can be performed by considering the infinitesimal parallelogram shown in Fig. B.3. Denoting by s_1 , s_2 and s_3 the distances of point P from the corresponding opposite sides and by h_1 , h_2 and h_3 the respective heights of the triangle, we can write

$$A = \frac{a_i h_i}{2}$$
 and $A_i = \frac{a_i s_i}{2}$ $(i = 1, 2, 3)$

which yield according to Eq. (B.27)

$$\frac{s_i}{h_i} = \frac{A_i}{A} = \xi_i$$

and hence

$$s_i = h_i \, \xi_i$$

So, the surface element dA can be expressed as

$$dA = \frac{ds_1 ds_2}{\sin \theta_3}$$
$$= \frac{(h_1 d\xi_1)(h_2 d\xi_2)}{\sin \theta_3}$$
$$= 2A d\xi_1 d\xi_2$$
(B.31)

Consequently, a domain integral may be written as

$$\int_{A} f(x,y) dA = 2A \int_{0}^{1} \int_{0}^{1-\xi_{1}} f(\xi_{1},\xi_{2},\xi_{3}) d\xi_{2} d\xi_{1}$$
$$= 2A \int_{0}^{1} \int_{0}^{1-\xi_{1}} f[\xi_{1},\xi_{2},(1-\xi_{1}-\xi_{2})] d\xi_{2} d\xi_{1}$$
(B.32)

If $f(\xi_1, \xi_2, \xi_3)$ is a polynomial term of the form $\xi_1^a \xi_2^b \xi_3^c$, where a, b and c are non-negative integers, then we have [6]

$$\int_{A} \xi_{1}^{a} \xi_{2}^{b} \xi_{3}^{c} dA = \frac{a! \ b! \ c!}{(a+b+c+2)!} 2A$$
(B.33)

A comprehensive introduction to triangular coordinates can be found in the book by Gallagher [7].

Gauss integration rules over triangles have been derived by means of triangular coordinates and have the form

$$I = \int_{0}^{1} \left[\int_{0}^{1-\xi_{1}} f(\xi_{1},\xi_{2},\xi_{3}) d\xi_{2} \right] d\xi_{1}$$

$$\approx \sum_{k=1}^{n} f(\xi_{1}^{k},\xi_{2}^{k},\xi_{3}^{k}) w_{k}$$
(B.34)

The coordinates ξ_1^k , ξ_2^k , ξ_3^k and the weight factors w_k for integration rules which are accurate for polynomials of order 1 to 5 are given in Table B.5. These values have been derived by Hammer *et al.* [8].

Number and position of the Gauss points	k	ξ_1^k	ξ_2^k	ξ_3^k	w_k / A
n = 1 Degree of accuracy: 1	1	1⁄3	¹ ⁄3	1⁄3	1
	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1⁄3
n=3	2	0	$\frac{1}{2}$	$\frac{1}{2}$	1⁄3
Degree of accuracy: 2	3	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$
	1	1/3	$\frac{1}{3}$	$\frac{1}{3}$	$-27/_{48}$
-X->	2	3⁄5	1⁄5	$\frac{1}{5}$	$25/_{48}$
n = 4	3	1/5	3⁄5	$\frac{1}{5}$	$25/_{48}$
Degree of accuracy: 3	4	1/5	1/5	3⁄5	$25/_{48}$
	1	0.33333333	0.33333333	0.33333333	0.22500000
	2	0.79742699	0.10128651	0.10128651	0.12593918
Nor and the second seco	3	0.10128651	0.79742699	0.10128651	0.12593918
1	4	0.10128651	0.10128651	0.79742699	0.12593918
n = 7	5	0.05971587	0.47014206	0.47014206	0.13239415
Degree of accuracy: 5	6	0.47014206	0.05971587	0.47014206	0.13239415
-	7	0.47014206	0.47014206	0.05971587	0.13239415

Table B.5 Coordinates and weights for Gauss integration over triangular domains.

Example B.2

The integrals below are computed using a 4-point Gaussian integration over the triangle shown in Fig. B.4.



Figure B.4 Triangular domain of integration.

Equations (B.29) applied to the triangle of Fig. B.4 become

$$x = 2\xi_1 + 3\xi_2 + \xi_3$$

$$y = \xi_1 + 2\xi_2 + 3\xi_3$$

and formula (B.34) yields

$$\begin{split} \int_{A} (1) dA &= \sum_{k=1}^{4} (1)_{k} w_{k} \\ &= (1) \left(-\frac{27A}{48} \right) + (1) \left(\frac{25A}{48} \right) + (1) \left(\frac{25A}{48} \right) + (1) \left(\frac{25A}{48} \right) = A \\ \int_{A} x \, dA &= \sum_{k=1}^{4} \left(2\xi_{1}^{k} + 3\xi_{2}^{k} + \xi_{3}^{k} \right) w_{k} \\ &= \left(2 \times \frac{1}{3} + 3 \times \frac{1}{3} + \frac{1}{3} \right) \left(-\frac{27}{48}A \right) + \left(2 \times \frac{3}{5} + 3 \times \frac{1}{5} + \frac{1}{5} \right) \left(\frac{25}{48}A \right) \\ &+ \left(2 \times \frac{1}{5} + 3 \times \frac{3}{5} + \frac{1}{5} \right) \left(\frac{25}{48}A \right) + \left(2 \times \frac{1}{5} + 3 \times \frac{1}{5} + \frac{3}{5} \right) \left(\frac{25}{48}A \right) \\ &= 2A \end{split}$$

$$\begin{split} \int_{A} y^{2} \, dA &= \sum_{k=1}^{4} \left(\xi_{1}^{k} + 2\xi_{2}^{k} + 3\xi_{3}^{k} \right)^{2} w_{k} \\ &= \left(\frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} \right)^{2} \left(-\frac{27}{48} A \right) + \left(\frac{3}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} \right)^{2} \left(\frac{25}{48} A \right) \\ &+ \left(\frac{1}{5} + 2 \times \frac{3}{5} + 3 \times \frac{1}{5} \right)^{2} \left(\frac{25}{48} A \right) + \left(\frac{1}{5} + 2 \times \frac{1}{5} + 3 \times \frac{3}{5} \right)^{2} \left(\frac{25}{48} A \right) \\ &= \frac{25}{6} A \end{split}$$

The computed values are exact because the 4-point integration rule is exact for polynomials up to the third degree.

B.4 Double singular integrals

The domain discretization method to compute the integrals (4.30) for Poisson's equation or (7.105) and (7.116) for plane elasticity problems requires the evaluation of double integrals, whose integrand behaves like lnr, 1/r or $1/r^2$. When the integration is performed over the element on which the field point lies, these integrals become singular or hyper-singular. Several special methods have been developed for their evaluation [9]. The method presented here is relatively simple and effective. It was developed in [10] for the fundamental solution of the Laplace equation and here it is extended also to that of Navier's equations.

B.4.1 Domain integrals of the fundamental solution for the Laplace equation

Consider the domain integral on the e-th element

$$\int_{\Omega^r} v \, d\Omega \tag{B.35}$$

where v(r) is the fundamental solution for the Laplace equation

$$v = \frac{1}{2\pi} \ell nr, \quad r = |Q - P| \quad (P, Q \in \Omega^c)$$
(B.36)

The domain integral (B.35) can be evaluated by converting it into a regular line integral on the boundary Γ^{e} . This is achieved using Green's identity

$$\int_{\Omega^{r}} \left(u \nabla^{2} U - U \nabla^{2} u \right) d\Omega = \int_{\Gamma^{r}} \left(u \frac{\partial U}{\partial n} - U \frac{\partial u}{\partial n} \right) ds$$
(B.37)

which for functions u and U defined as

$$u = 1 \quad \text{and} \quad \nabla^2 U = v^* \tag{B.38}$$

where

$$v^* = \frac{1}{2\pi} (\ell n r + 1) \tag{B.39}$$

yields

$$\int_{\Omega^r} v^* d\Omega = \int_{\Gamma^r} \frac{\partial U}{\partial n} \, ds \tag{B.40}$$

Combining Eqs. (B.36) and (B.39), we find that

$$\int_{\Omega^{r}} v \, d\Omega = \int_{\Omega^{r}} v^{\star} \, d\Omega - \frac{A^{c}}{2\pi}$$
$$= \int_{\Gamma^{r}} U_{n} \, ds - \frac{A^{c}}{2\pi}$$
(B.41)

where A^{e} is the area of the e-th element.

The function U may be established by expressing the second of Eqs. (B.38) in polar coordinates,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) = \frac{1}{2\pi}(\ell nr + 1)$$

which gives after two consecutive integrations

$$U = \frac{1}{8\pi} r^2 \ell n r \tag{B.42}$$

In the case that the domain integral involves derivatives of v, the function U in Eq. (B.41) is replaced with the respective derivatives. Namely,

$$\int_{\Omega'} v_m d\Omega = \int_{\Gamma'} U_{mn} ds \qquad (m = x, y, xx, xy, yy)$$
(B.43)

where subscripts preceded by a comma denote differentiation.

The derivatives U_{n} and U_{mn} are evaluated from the relations

$$U_{,n} = \frac{1}{8\pi} (2\ell nr + 1) r r_{,n}$$
(B.44a)

$$U_{,m} = -\frac{1}{8\pi} \Big[(2\ell nr + 1)\cos(\alpha + \phi) + 2\cos\alpha\cos\phi \Big]$$
(B.44b)

$$U_{yn} = -\frac{1}{8\pi} \Big[(2\ell nr + 1)\sin(\alpha + \phi) + 2\sin\alpha\cos\phi \Big]$$
(B.44c)

$$U_{,xxn} = \frac{1}{4\pi r} \Big(\cos\phi - \sin 2\alpha \,\sin\phi\Big) \tag{B.44d}$$

$$U_{,xyn} = \frac{1}{4\pi r} \cos 2\alpha \, \sin \phi \tag{B.44e}$$

$$U_{,yyn} = \frac{1}{4\pi r} \Big(\cos\phi + \sin 2\alpha \, \sin\phi\Big) \tag{B.44f}$$

where $\alpha = \text{angle}(\mathbf{x}, \mathbf{r})$ and $\phi = \text{angle}(\mathbf{r}, \mathbf{n})$ (see Fig. A.1).

B.4.2 Domain integrals of the fundamental solution for the Navier equations

Consider the domain integral on the e-th element

$$\int_{\Omega'} v^* d\Omega \tag{B.45}$$

where

$$v^* = U_{\xi x}$$
 or $U_{\eta x}$ or $U_{\xi y}$ or $U_{\eta y}$ (B.46)

The integral of Eq. (B.45) can be evaluated by converting it into a regular line integral on the boundary Γ^{e} . This is achieved using Eqs. (7.139) and (7.140). For example, choosing $V = \xi$ as the potential function in Eqs. (7.135) and (7.139), results in

$$\int_{\Omega^{r}} U_{\xi x} d\Omega = \frac{2(1-\overline{\nu})}{8\pi G} \int_{\Gamma^{r}} \left[\xi \left(\frac{\partial \phi_{,\xi}}{\partial n} \right) - \phi_{,\xi} n_{x} \right] ds + \int_{\Gamma^{r}} \left(U_{\xi x} n_{x} + U_{\eta x} n_{y} \right) ds$$
(B.47)

The domain integrals of the kernels $T_{\xi x}$, $T_{\eta x}$, $T_{\xi y}$, $T_{\eta y}$, $\sigma_{x\xi}$, $\sigma_{x\eta}$, ..., $\tau_{xy\eta}$, $\overline{\sigma}_{x\xi}$, $\overline{\sigma}_{x\eta}$, ..., $\overline{\tau}_{xy\eta}$ are expressed in terms of the derivatives of (B.45), and are evaluated after they have been converted to line integrals.

B.5 References

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Appendix C

Answers to selected problems

Chapter 2

2.1. (i) $\int_{\Gamma} \frac{x^2}{2} n_x ds$ or $\int_{\Gamma} xy n_y ds$ (ii) $\int_{\Gamma} xy n_x ds$ or $\int_{\Gamma} \frac{y^2}{2} n_y ds$ (iii) $\int_{\Gamma} \frac{x^2 y}{2} n_x ds$ or $\int_{\Gamma} \frac{xy^2}{2} n_y ds$ (iv) $\int_{\Gamma} \frac{x^3}{3} n_x ds$ or $\int_{\Gamma} x^2 y n_y ds$ (v) $\int_{\Gamma} xy^2 n_x ds$ or $\int_{\Gamma} \frac{y^3}{3} n_y ds$ (vi) $\int_{\Gamma} \frac{1}{3} (x^3 n_x + y^3 n_y) ds$ or $\int_{\Gamma} xy (y n_x + x n_y) ds$ (vii) $\int_{\Gamma} \sin x n_x ds$ or $\int_{\Gamma} y \cos x n_y ds$

2.2. Hint:

- (a) Use the transformation relations from Cartesian to polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$
- (b) Apply the chain rule of differentiation to express the derivatives in polar coordinates
$$\frac{\partial}{\partial x} = r_{,x} \frac{\partial}{\partial r} + \theta_{,x} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = r_{,y} \frac{\partial}{\partial r} + \theta_{,y} \frac{\partial}{\partial \theta}$$
$$r_{,x} = \cos\theta, \quad r_{,y} = \sin\theta, \quad \theta_{,x} = -\frac{\sin\theta}{r}, \quad \theta_{,y} = \frac{\cos\theta}{r}$$

(c) Repeat the differentiation to obtain the second derivatives and by adding them derive the Laplacian in polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

which for the axisymmetric case, v = v(r), becomes

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

(d) Establish a particular solution of the equation

$$\nabla^2 U = \ell n r$$

(e) Apply Green's identity for u = 1 and U to obtain

$$\int_{\Omega} \ell nr \, d\Omega = \int_{\Gamma} \frac{\partial U}{\partial n} ds \,, \text{ where } U = \frac{1}{4} r^2 (\ell nr - 1)$$

2.3. (i)
$$\int_{a}^{b} \delta(x - x_{0}) dx = 1$$

(ii) $\int_{a}^{b} \delta(kx) f(x) dx = \frac{f(0)}{|k|}$ (*hint:* use appropriately Eq. 2.40)
(iii) $\int_{a}^{b} \delta(-x) dx = 1$
(iv) $\int_{a}^{b} \delta^{(n)}(x) \phi(x) dx = (-1)^{n} \phi^{(n)}(x_{0})$

- 2.4. Hint: Use appropriately Eq. (2.40).
- **2.5.** Hint: Apply Eqs. (2.40) for $x = r \cos \theta$, $y = r \sin \theta$ to obtain

$$\delta(P - P_0) = \delta(x - x_0)\delta(y - y_0) = \frac{\delta(r - r_0)\delta(\theta - \theta_0)}{r}$$

2.6. *Hint:* Apply repeatedly the Gauss-Green theorem for all terms in the integral $\int_{\Omega} v L(u) d\Omega$

until all derivatives of u are eliminated and group appropriately the boundary terms.

2.7. *Hint*: Apply Green's identity for u and v = 1.

2.8. (iii)
$$\int_{\Omega} \left[vL(u) - uL^{*}(v) \right] d\Omega = \int_{\Gamma} \left[v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} + uv(\mathbf{a} \cdot \mathbf{n}) \right] ds$$

Chapter 3

3.1.
$$u = \frac{1}{12}xy(x^2y + xy^2)$$

- **3.2.** $\int_{\Omega} vf \, d\Omega = \frac{1}{4} R^2 (2\ell nR 1) f(x_o, y_o), \quad f(x_o, y_o) = \alpha_0 + \alpha_1 x_o + \alpha_2 y_o$
- **3.3.** (i) *Hint:* use Eq. (3.49) and differentiate it with respect to x. See also Section B.4 of Appendix B.
 - (ii) *Hint:* Establish a function F satisfying the equation $\nabla^2 F = x^2 + y^2$

Then apply Green's identity for $v = lnr/2\pi$ and u = F, differentiate it with respect to x and use Eq. (2.42).

3.4. *Hint:* Show first that the integral representation (3.18) can be written as

$$u(P) = -\int_{\Gamma} \left\{ v(P,q) \frac{\partial u(q)}{\partial n_q} - \left[u(q) - u(p) \right] \frac{\partial v(P,q)}{\partial n_q} \right\} ds_q + u(p)$$

where p is a point on Γ . Then, to obtain $\partial u/\partial n_p$, differentiate with respect to a direction n and find the limit as $P \rightarrow p \in \Gamma$ choosing n to coincide with the normal to the boundary at point $p \in \Gamma$. The limiting procedure is similar to that of Section 3.3. Work in the same way to obtain $\partial u/\partial t_p$. For points p where the boundary is smooth, we find

$$\frac{1}{2}\frac{\partial u(p)}{\partial n_p} = -\int_{\Gamma} \left\{ \frac{\partial v(p,q)}{\partial n_p} \frac{\partial u(q)}{\partial n_q} - [u(q) - u(p)] \frac{\partial^2 v(p,q)}{\partial n_p \partial n_q} \right\} ds_q$$
$$\frac{1}{2}\frac{\partial u(p)}{\partial t_p} = -\int_{\Gamma} \left\{ \frac{\partial v(p,q)}{\partial t_p} \frac{\partial u(q)}{\partial n_q} - [u(q) - u(p)] \frac{\partial^2 v(p,q)}{\partial t_p \partial n_q} \right\} ds_q$$

3.5. *Hint:* For $p \in \Gamma$ use the integral representations of $\partial u/\partial n_p$ and $\partial u/\partial t_p$ to derive the expressions for $\partial u/\partial x$ and $\partial u/\partial y$.

Chapter 4

4.1. Hint: See subroutine DERIV of program FLUIDCON in Chapter 6.

4.3. Hint:

(a) Evaluate the matrices $[H]^1$, $[G]^1$, $[H]^2$ and $[G]^2$.

Domain Ω_1 is isotropic. The boundary integral equation is given by Eq. (3.29), which after discretization yields

$$[H]^1 \{u\}^1 = [G]^1 \{q_n\}^1, \ \ \{q_n\}^1 = \{u_{,n}\}^1$$
 ($u_{,n} =
abla u \cdot \mathbf{n}$)

The matrices $[H]^1$, $[G]^1$ are evaluated as described in Section (4.3).

Domain Ω_2 is orthotropic. The boundary integral equation is given by Eq. (3.79), which after discretization yields

$$[H]^{2}{\{u\}}^{2} = [G]^{2}{\{q_{n}\}}^{2}, \ {\{q_{n}\}}^{2} = {\{u_{,m}\}}^{2} \quad (u_{,m} = \nabla u \cdot \mathbf{m})$$

where

$$H_{ij} = \int_{\Gamma_j} \frac{1}{2\pi\sqrt{|D|}} \frac{r_m}{r} ds_q - \frac{1}{2}\delta_{ij}$$
$$G_{ij} = \int_{\Gamma_j} \frac{1}{2\pi\sqrt{|D|}} \ell nr ds_q$$

in which

$$r(q, p_i) = \sqrt{\frac{(\xi - x_i)^2}{k_{xx}} + \frac{(\eta - y_i)^2}{k_{yy}}}$$

$$q:(\xi,\eta)\in \Gamma_j$$
 and $p_i:(x_i,y_i)\in \Gamma_i$.

The elements of these matrices can be evaluated in a way similar to that presented in Section 4.3.

(b) Apply the given boundary conditions on the outer boundaries and the interface continuity conditions to obtain the matrices [A] and $\{B\}$ as presented in Section 4.9.

Chapter 5

5.1. (i)
$$\psi_1 = -\frac{1}{16} + \frac{1}{16}\xi + \frac{9}{16}\xi^2 - \frac{9}{16}\xi^3$$
, $\psi_2 = \frac{9}{16} - \frac{27}{16}\xi - \frac{9}{16}\xi^2 + \frac{27}{16}\xi^3$,
 $\psi_3 = \frac{9}{16} + \frac{27}{16}\xi - \frac{9}{16}\xi^2 - \frac{27}{16}\xi^3$, $\psi_4 = -\frac{1}{16} - \frac{1}{16}\xi + \frac{9}{16}\xi^2 + \frac{9}{16}\xi^3$

(ii)
$$\psi_1 = -\frac{1}{6} + \frac{1}{6}\xi + \frac{2}{3}\xi^2 - \frac{2}{3}\xi^3$$
, $\psi_2 = \frac{2}{3} - \frac{4}{3}\xi - \frac{2}{3}\xi^2 + \frac{4}{3}\xi^3$,
 $\psi_3 = \frac{2}{3} + \frac{4}{3}\xi - \frac{2}{3}\xi^2 - \frac{4}{3}\xi^3$, $\psi_4 = -\frac{1}{6} - \frac{1}{6}\xi + \frac{2}{3}\xi^2 + \frac{2}{3}\xi^3$

5.2. The exact value for the area of the circular sector is given as

$$A_{\text{exact}} = \frac{\theta_0}{2} R^2 \Rightarrow A_{\text{exact}} = \frac{3\pi}{8}$$

(i) For a linear element approximation: $A_{\text{linear}} = \frac{1}{2}R^2 \sin \theta_0$

- (ii) For a quadratic element approximation follow the steps:
 - (a) Use polar coordinates to evaluate the area

$$A = \frac{1}{2} \int_0^{\theta_0} r^2 \, d\theta$$

(b) Transform the integral on the interval $-1 \le \xi \le 1$

$$A = \frac{1}{2} \int_{-1}^{+1} \sqrt{x(\xi)^2 + y(\xi)^2} |J(\xi)| d\xi$$

where $x(\xi)$ and $y(\xi)$ are obtained from Eqs. (5.66) and the Jacobian $|J(\xi)|$ from Eq. (5.68).

- (c) Evaluate the integral using a four-point Gauss integration.
- (iii) For a cubic element approximation establish first

$$x(\xi) = \sum_{k=1}^{4} x_k \psi_k(\xi)$$
 and $y(\xi) = \sum_{k=1}^{4} y_k \psi_k(\xi)$

from the known coordinates of the four nodal points, which are

$$x_k = R \cos\left[(k-1)\frac{\theta_0}{3}\right]$$
 and $y_k = R \sin\left[(k-1)\frac{\theta_0}{3}\right]$

where k = 1, 2, 3, 4. The shape functions $\psi_k(\xi)$ are those derived in Problem 5.1-(i). Then, follow the same procedure as in case (ii) to evaluate the area integral.

The exact value of the area along with the computed values for the three types of elements are given in the following table.

	Exact	Linear	Parabolic	Cubic
Area	1.178097	1.164686	1.178086	1.178099
Error (%)	0	1.14	9.34×10^{-6}	-1.70×10^{-6}

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5.3. The integrand has a peak at x = 0.25 (see its graph in the figure). The interval is divided into subintervals as in the table and Gauss integration is applied on each of them

Number of subintervals	Subintervals	Number of Gauss points	$\int_{-1}^{+1} \frac{dx}{\left[(x-0.25)^2+0.05\right]^4}$
4	[-1.00, +0.10]	8	35124.17
	[+0.10, +0.25]	6	
	[+0.25, +0.40]	6	
	[+0.40, +1.00]	8	
Exact value			35123.22
1600 1200	00		
800	00		
400	00		
	0 -1 -0.8 -0.6 -0	.4-0.2 0 0.2	0.4 0.6 0.8 1

- **5.4.** *Hint:* Transform the integral on the interval $-1 \le \xi \le +1$ and use Gauss integration to evaluate it. It is suggested to follow the steps:
 - (a) Express the coordinates of the points on the parabolic element in terms of ξ using Eq. (5.63), namely

$$x(\xi) = 4.30\,\psi_1(\xi) + 4.10\,\psi_2(\xi) + 3.80\,\psi_3(\xi)$$

$$y(\xi) = 2.50 \,\psi_1(\xi) + 2.90 \,\psi_2(\xi) + 3.20 \,\psi_3(\xi)$$

(b) Find the expressions of $r(\xi)$ and $|J(\xi)|$ to obtain the integrand in the form

$$f(\xi) = \psi_1(\xi) \, \ell n [r(\xi)] \, |J(\xi)|$$

= $-\frac{1}{2} \xi (1-\xi) \, \ell n \left\{ \sqrt{[x(\xi)-4.15]^2 + [y(\xi)-2.65]^2} \right\} \times \sqrt{[x'(\xi)]^2 + [y'(\xi)]^2}$

- (c) Check whether the integral is singular or near-singular. For this purpose evaluate $\xi = \xi_0$ for which $r(\xi)$ becomes minimum as well as the values of $r(\xi_0)$ and $f(\xi_0)$ (answer: $\xi_0 = -0.56595534$, $r(\xi_0) = 0.08357687$, $f(\xi_0) = -0.49520518$). Hence the integral is neither singular nor near-singular. This also becomes evident from the graph of the integrand shown in the figure.
- (d) Evaluate the integral using Gauss integration

$$\int_{-1}^{+1} f(\xi) d\xi \simeq \sum_{k=1}^{n} f(\xi_k) w_k = I_n \Rightarrow \begin{cases} I_4 = -0.34181408 \\ I_6 = -0.35966759 \\ I_8 = -0.35957404 \end{cases}$$



5.5. Hint:

(a) Evaluate the length of the j-th element,

$$\ell_{j} = \frac{2}{\kappa} \sqrt{(x_{2} - x_{1})^{2} + (y_{2} - y_{1})^{2}} = 1.166190$$

- (b) Use Eq. (5.27) together with Eqs. (5.32) and (5.35) to evaluate g_1^{ij} .
- (c) Use Eq. (5.36) together with Eqs. (5.37) and (5.38) to evaluate g_2^{ij} .

The influence coefficients, when the source node i coincides with nodes 1 and 2 (local numbering) of the integration element j, are

for node-*i* at $\xi_1 = -0.5$: $g_1^{1j} = -0.215355$, $g_2^{1j} = -0.039519$ for node-*i* at $\xi_2 = +0.5$: $g_1^{2j} = -0.046087$, $g_2^{2j} = -0.184665$

Chapter 6

6.1. Use program TORSCON to evaluate the required quantities for the three cross-sections. The computed answers along with the exact values or those obtained with other approximate solutions are given below in tabular form.

Cross- section	Solution method	$\frac{I_t}{h^4}$	$\frac{\max \tau_{tz}}{G\theta h}$
	BEM (N=250)	0.028593	0.464960
	Analytical ⁽¹⁾	0.028585	0.465030
	BEM (N=250)	0.053527	0.779140
	Approximate ⁽²⁾	0.051157	
52	BEM (N=250)	0.053780	0.677690
	Approximate ⁽²⁾	0.057911	

⁽¹⁾ see Example 6.2

⁽²⁾ see Chapter 6, Ref. [19]

6.2. *Hint:* Seek the solution as a sum of the homogeneous solution and a particular one,

 $u = u_0 + u_1$

(a) The particular solution is obtained according to the procedure described in Section 3.4.2

$$\nabla^2 u_1 = -\frac{1}{30} x$$

which yields

$$u_1 = -\frac{x(x^2 + y^2)}{240}$$

(b) The homogeneous solution obtained from the boundary value problem

$$abla^2 u_0 = 0 \quad \text{in } \Omega$$
 $u_0 = rac{x(x^2 + y^2)}{240} \quad \text{on } \Gamma$

using the program LABECON.

Answer: u(a/2, b/3) = 0.018 m employing N=200 boundary elements.

- 6.3. *Hint:* Use program LABECONMU.
- **6.4.** *Hint:* At the insulated parts of the boundary it is $T_n = 0$.

Answer: Values of the temperature T at selected points on the axis of symmetry of the cross-section are given in the table below as obtained using LABECON with N=240.

y	0.05	0.25	0.45	0.65	0.85
Т	14.54	32.67	50.82	67.35	77.86

- **6.5.** *Hint:* Modify program LABECON, according to the procedure presented in Section 4.9 (method of subdomains), so that it can be used for the analysis of composite domains.
- 6.6. *Hint:* Use the program developed for the requirements of Problem 4.2.
- 6.7. *Hint:* Use program FLUIDCON.

Answer: Values of the velocity component v_n at selected points of the outlet cross-section are given in the following table as obtained using FLUIDCON with N= 240.

\overline{y}	1.70	1.30	0.90	0.50	0.10
v_n	0.484	0.492	0.503	0.512	0.516

Chapter 7

7.1. Given the initial strains $\{\varepsilon_0\}$ use Eq. (7.13) to establish the corresponding stresses $\{\sigma_0\}$. Then substitute Eq. (7.26) into the equilibrium equation (7.18) with $b_x = b_y = 0$ to obtain

$$\nabla^2 u + \frac{1}{1 - 2\nu} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \frac{1}{G} b_x^0 = 0$$

$$\nabla^2 v + \frac{1}{1 - 2\nu} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{1}{G} b_y^0 = 0$$

where

$$b_x^0 = -\left(rac{\partial \sigma_x^0}{\partial x} + rac{\partial \tau_{xy}^0}{\partial y}
ight)$$

 $b_y^0 = -\left(rac{\partial \tau_{xy}^0}{\partial x} + rac{\partial \sigma_y^0}{\partial y}
ight)$

The boundary tractions due to the initial strains are obtained by substituting Eq. (7.26) into Eqs. (7.22) and taking into account that

$$t_x = t_x^t - t_x^0 = 0$$
$$t_y = t_y^t - t_y^0 = 0$$

where t'_x and t'_y denote the total boundary tractions. The derived expressions are

$$t_x^t = t_x^0 = -(\sigma_x^0 n_x + \tau_{xy}^0 n_y)$$
$$t_y' = t_y^0 = -(\tau_{xy}^0 n_x + \sigma_y^0 n_y)$$

Therefore, the displacement field u_0 and v_0 due to initial strains is the one produced by the effective body forces b_x^0 , b_y^0 in Ω and the prescribed boundary tractions t_x^0 and t_y^0 on Γ . In this case, the boundary integral equations result from Eqs. (7.90) by replacing b_x , b_y and t_x , t_y with b_x^0 , b_y^0 and t_x^0 , t_y^0 , respectively. In solving this problem using program ELBECON, attention should be paid in restraining the rigid body motion, so that the matrix [H]can be inverted.

7.2. The initial stress distribution caused by the temperature change $\Delta T(x, y)$ is given by Eq. (7.28) and the effective body forces (see answer for Problem 7.1) become

$$b_x^0 = -\frac{\overline{E} \ \alpha}{1 - \overline{\nu}} \ \frac{\partial \Delta T}{\partial x} = 0$$
 and $b_y^0 = -\frac{\overline{E} \ \alpha}{1 - \overline{\nu}} \ \frac{\partial \Delta T}{\partial y} = 0$

while the boundary tractions become

$$t_x^0 = \frac{\overline{E}\alpha\Delta T}{1-\overline{\nu}}n_x = 222.22\,n_x$$
 and $t_y^0 = \frac{\overline{E}\alpha\Delta T}{1-\overline{\nu}}n_y = 222.22\,n_y$

The solution of the problem is obtained following three steps:

(a) Evaluation of the displacements produced by the temperature change. Due to the double symmetry of the cross-section the solution can be limited to the lower left quadrant (see figure). This problem is solved using program ELBECON with the boundary conditions shown in the figure.



(b) Solve the problem for the whole domain with the following boundary conditions:

outer boundary:	$u(0,y) = -u^0(0,y),$	$t_y(0,y)=0$
	$v(x,0) = -v^0(x,0),$	$t_x(x,0)=0$
	$u(2.5, y) = -u^0(2.5, y),$	$t_y(2.5,y)=0$
	$t_x(x,1.5)=0,$	$t_y(x,1.5)=0$

inner boundary: $t_x = 0$, $t_y = 0$ (traction-free)

(c) Superimpose the solutions of steps (a) and (b).

Numerical results obtained for N=348 boundary elements are given in the following table.

	<i>x</i> =	= 0		x = 0.425			
	$v \times 10^3$	t_x	$u \times 10^3$	$v \times 10^3$	σ_x	σ_y	$ au_{xy}$
0.3125	0.366	145.71	0.040	0.371	-166.29	15.90	21.50
0.5125	0.577	120.34	0.115	0.595	-118.07	2.93	40.49
0.9875	1.018	119.13	0.126	-0.616	-111.11	-1.93	-27.24
1.1875	1.216	150.09	0.043	-0.477	-164.88	8.29	-12.87

7.3. Hint:

(a) Establish the displacement and stress fields for temperature change working in two stages as in Problem 7.2, including, however, the body forces

$$b_x^0 = -\frac{\overline{E} \ \alpha}{1 - \overline{\nu}} \frac{\partial T}{\partial x}, \qquad b_y^0 = -\frac{\overline{E} \ \alpha}{1 - \overline{\nu}} \frac{\partial T}{\partial y}$$

which do not vanish in this case. Nevertheless, the domain integrals involving the body forces can be converted to boundary line integrals by means of Eqs. (7.139) and (7.140), since they are derived from a potential T ($\nabla^2 T = 0$). For the needs of this problem, program ELBECON must be modified to include the vector $\{F\}$ of Eq. (7.111) resulting from the presence of body forces.

- (b) Superimpose the solution of part (a) to that for internal pressure in order to obtain the total deformation and stresses in the pipe.
- **7.4.** *Hint:* Determine a particular solution u_1 , v_1 using the relevant expressions derived in Example 7.1. Then employ program ELBECON after modifying appropriately the boundary conditions as indicated in Eqs. (7.124).
- 7.5. Answer:

$$u_G = u_H = 7.3699 \times 10^{-3} \text{ m}$$

$$K_{el} = \frac{P_{total}}{u_G} = \frac{2 \times 750 \text{ kN/m} \times 0.60 \text{ m}}{7.3699 \times 10^{-3} \text{ m}} = 1.22118 \times 10^5 \text{ kN/m}$$

It is worth mentioning that, modeling the given frame with three beam elements (accounting also for shear deformation) and considering that the frame is clamped at the bottom of the two columns, the lateral displacement and corresponding stiffness are found to be

$$u_{bcam} = 7.6 \times 10^{-3} \text{ m}$$
$$K_{bcam} = 1.18421 \times 10^5 \text{ kN/m}$$

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